Existence and stability of a solution with a new prescribed behavior for a heat equation with a critical nonlinear gradient term

Hatem ZAAG
CNRS and LAGA Université Paris 13
Equadiff 2015
Lyon, July 6-10, 2015

Joint work with:
S. Tayachi (Faculté des Sciences de Tunis).
**Introduction: The equation**

We consider the following PDE:

\[
\begin{align*}
\partial_t u &= \Delta u + \mu |\nabla u|^q + |u|^{p-1}u, \\
u(\cdot,0) &= u_0
\end{align*}
\]

where:

- $p > 3$, $\mu > 0$, $q = q_c = \frac{2p}{p+1}$,
- $u(t): x \in \mathbb{R}^N \rightarrow u(x,t) \in \mathbb{R}$,
- $u_0 \in W^{1,\infty}(\mathbb{R}^N)$. 

Hatem ZAAG (P13 & CNRS)

Existence and stability of a solution for a heat equation with a critical nonlinear gradient term

Equadiff2015 Lyon, July 6-10, 2015
History of the equation

- **Introduction**: Chipot-Weissler (1989), mathematical motivation ($\mu < 0$).


- **Mathematical analysis**: Chipot, Weissler, Peletier, Kawohl, Fila, Quittner, Deng, Alfonsi, Tayachi, Souplet, Snoussi, Galaktionov, Vázquez, Ebde, Z., Nguyen, ...

- **Elliptic version**: Chipot, Weissler, Serrin, Zou, Peletier, Voirol, Fila, Quittner, Bandle ...
Two limiting cases

- When $\mu = 0$, this is the well-known semilinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1}u.$$ 

- When $\mu = +\infty$, we recover (after rescaling) the Diffusive Hamilton-Jacobi equation:

$$\partial_t u = \Delta u + |\nabla u|^q.$$
This is a critical case

When \( \mu \in \mathbb{R} \) and \( w(y, s) \) is the similarity variables version of \( u(x, t) \):

\[
w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T - t}} \quad \text{and} \quad s = -\log(T - t),
\]

we have for all \( s \geq -\log T \) and \( y \in \mathbb{R}^N \):

\[
\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.
\]

with

\[
\alpha = \frac{q(p+1)}{2(p-1)} - \frac{p}{p-1} = \frac{(p+1)}{2(p-1)} (q - q_c) \quad \text{and} \quad q_c = \frac{2p}{p+1}.
\]

Therefore, we have 3 cases:

- **subcritical** when \( q < q_c \): we have a “perturbation” of the semilinear heat equation;

- **supercritical** when \( q > q_c \): we are in the Hamilton-Jacobi limit;

- **critical** when \( q = q_c \): this is the aim of the talk.
Other indications for the criticality of $q_c = 2p/(p + 1)$

- **Scaling:** Only when $q = q_c$, we have “$u$ solution $\Rightarrow u_\lambda$ solution”, where
  \[ u_\lambda(x, t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t), \ \forall \lambda > 0, \ \forall t > 0, \ x \in \mathbb{R}^N, \]
  as for the equation without gradient term ($\mu = 0$).

- **Large time behavior:** it depends on whether $q < q_c$, $q = q_c$, $q > q_c$; see Snoussi-Tayachi-Weissler (1999) and Snoussi-Tayachi (2007).

- **Blow-up behavior for $\mu < 0$:** it depends also on whether $q < q_c$, $q = q_c$, $q > q_c$; see Souplet (2001, 2005), Chlebik, Fila and Quittner (2003) (Bounded Domain)

- Also for the elliptic version.
Cauchy problem and blow-up solutions

- **Cauchy problem**: Wellposed in $W^{1,\infty}(\mathbb{R}^N)$ (fixed point argument, see Alfonsi-Weissler (1993), Souplet-Weissler (1999)).

- **Blow-up solutions**: If $T < \infty$, then $\lim_{t \to T} \|u(t)\|_{W^{1,\infty}(\mathbb{R}^N)} = \infty$.

**Definition**: $x_0$ is a blow-up point if $\exists (t_n, x_n) \to (T, x_0)$ s.t. $|u(x, t)| \to \infty$ as $n \to \infty$. 
Aim of the talk

Take \( q = q_c \).

We have 3 goals:

- construct a blow-up solution,

- determine its blow-up profile,

- prove its stability (with respect to perturbations in initial data).
Contents

1 The new blow-up profile
   • History of the problem \((q \leq q_c)\)
   • Existence of the new profile \((q = q_c)\)
   • The stability result
The new blow-up profile
- History of the problem \((q \leq q_c)\)
- Existence of the new profile \((q = q_c)\)
- The stability result

The proofs
- A formal approach for the existence result
- A sketch of the proof of the existence result
- Proof of the stability result
Contents

1 The new blow-up profile
   - History of the problem \((q \leq q_c)\)
   - Existence of the new profile \((q = q_c)\)
   - The stability result

2 The proofs
The new blow-up profile

History of the problem ($q \leq q_c$)

Case $\mu = 0$: the standard semilinear heat equation

- The (generic) profile is given by

\[(T - t)^{1/(p-1)}u(z\sqrt{(T - t)|\log(T - t)|}, t) \sim f_0(z) \text{ as } t \to T,\]

where

\[f_0(x) = (p - 1 + b_0|x|^2)^{-1/(p-1)} \text{ and } b_0 = (p - 1)^2/(4p).\]


- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

- Other profiles are possible.
Ebde and Z. (2011) could adapt the previous existence strategy and find the same behavior as for $\mu = 0$, since the gradient term is subcritical in size in similarity variables:

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w + \mu e^{\alpha s} |\nabla w|^q.$$  

with

$$\alpha = \frac{(p+1)}{2(p-1)} (q - q_c) < 0.$$
Critical case: $q = q_c$, with $-2 < \mu < 0$ and $p - 1 > 0$ small

Exact self-similar blow-up solution by Souplet, Tayachi and Weissler (1996):

$$u(x, t) = (T - t)^{-1/(p-1)} W \left( \frac{|x|}{\sqrt{T - t}} \right)$$

where $W$ satisfies the following elliptic equation:

$$W'' + \frac{N - 1}{r} W' - \frac{1}{2} r W' - \frac{W}{p - 1} + W^p + \mu |W'|^{q_c} = 0.$$
Critical case: \( q = q_c \); A numerical result

A similar profile to the case \( \mu = 0 \) was discovered \textit{numerically} by Van Tien Nguyen (2014):

\[
(T - t)^{1/(p-1)} u(z \sqrt{(T - t) \log(T - t)}), t) \sim f_0(z) \text{ as } t \to T,
\]

where

\[
f_\mu(x) = \left( p - 1 + b_\mu |x|^2 \right)^{-1/(p-1)}
\]

with

\[
b_\mu > 0 \text{ and } b_0 = (p - 1)^2 / (4p),
\]

the same as for \( \mu = 0 \).

\textbf{Remark}: We initially wanted to confirm this result, and ended by finding a \textit{new} type of behavior.
The new blow-up profile

Existence of the new profile \((q = q_c)\)

Critical case: \(q = q_c\); Our new profile

**Theorem (Tayachi and Z.)** There exists a solution \(u(x, t)\) s.t.:

- **Simultaneous Blow-up:** Both \(u\) and \(\nabla u\) blow up as \(t \to T > 0\) only at the origin;

- **Blow-up Profile:**

  \[
  (T - t)^{\frac{1}{p-1}} u(z \sqrt{T - t} |\log(T - t)|^{\frac{p+1}{2(p-1)}}, t) \sim f_{\mu}(z) \text{ as } t \to T
  \]

  with

  \[
  f_{\mu}(z) = (p - 1 + \bar{b}_\mu |z|^2)^{-\frac{1}{p-1}} \quad \text{with} \quad \bar{b}_\mu = \frac{1}{2(p-1)} \left( p - 2 \right) \frac{p-2}{p-1} \left( \frac{2\pi}{p} \int_{\mathbb{R}^N} |y| q e^{-|y|^2/4} dy \right)^{\frac{p+1}{p-1}} \mu^{\frac{p+1}{p-1}} > 0.
  \]

- **Final profile** When \(x \neq 0\), \(u(x, t) \to u(x, T)\) as \(t \to T\) with

  \[
  u(x, T) \sim \left( \frac{\bar{b}_\mu |x|^2}{2 |\log |x||^{\frac{p+1}{p-1}}} \right)^{-\frac{1}{p-1}} \quad \text{as} \quad x \to 0.
  \]
The exhibited behavior is new in two respects:

- **The scaling law**: $\sqrt{T - t} |\log(T - t)|^{\frac{p+1}{2(p-1)}}$ instead of the laws of the case $\mu = 0$, $\sqrt{(T - t)|\log(T - t)|}$ or $(T - t)^{\frac{1}{2m}}$ where $m \geq 2$ is an integer;

- **The profile function**: $\bar{f}_\mu(z) = (p - 1 + \bar{b}_\mu|z|^2)^{-\frac{1}{p-1}}$ is different from the profile of the case $\mu = 0$, namely $f_0(z) = (p - 1 + b_0|z|^2)^{-\frac{1}{p-1}}$, in the sense that $\bar{b}_\mu \neq b_0$.

Note in particular, that

$$\bar{b}_\mu \to \infty \text{ as } \mu \to 0.$$ 

**Remark**: Our solution is different already in the scaling from the numerical solution of Van Tien Nguyen, which is in the $\mu = 0$ style.
Idea of the proof

We follow the constructive existence proof used by Bricmont-Kupiainen (1994), Merle-Z. (1997) for the standard semilinear heat equation.

That method is based on:

- The reduction of the problem to a finite-dimensional one \((N + 1)\) parameters;
- The solution of the finite-dimensional problem thanks to the degree theory.
Critical case: $q = q_c$; Stability of the constructed solution

Thanks to the interpretation of the $(N + 1)$ parameters of the finite-dimensional problem in terms of the blow-up time (in $\mathbb{R}$) and the blow-up point (in $\mathbb{R}^N$), the existence proof yields the following:

**Theorem (Tayachi and Z.: Stability)**

*The constructed solution is stable with respect to perturbations in initial data in $W^{1,\infty}(\mathbb{R}^N)$.*
Corollary (Tayachi and Z.)

After an appropriate scaling, our results yield stable blow-up solutions for the following Viscous Hamilton-Jacobi equation:

$$\partial_t v = \Delta v + |\nabla v|^q + \nu |v|^{p-1} v;$$

with

$$\nu > 0, \ 3/2 < q < 2, \ p = \frac{q}{2 - q}.$$ 

The solution and its gradient blow up simultaneously, only at one point. Of course, the blow-up profile is given after an appropriate scaling.
Contents

1 The new blow-up profile

2 The proofs
   • A formal approach for the existence result
   • A sketch of the proof of the existence result
   • Proof of the stability result
A formal approach to find the ansatz ($N = 1$)

Following the standard semilinear heat equation case, we work in similarity variables:

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T - t}} \quad \text{and} \quad s = -\log(T - t).$$

We need to find a solution for the following equation defined for all $s \geq s_0$ and $y \in \mathbb{R}^N$:

$$\partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^q,$$

such that

$$0 < \epsilon_0 \leq \|w(s)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\epsilon_0} \quad (\text{type 1 blow-up}).$$

Idea 1: Look for a (non trivial) stationary solution: already successful by Souplet, Tayachi and Weissler (1996), for $p$ close to 1 (self-similar solution in the $u(x, t)$ setting).

Idea 2: Since $w \equiv \kappa \equiv (p - 1)^{-\frac{1}{p-1}}$ is a trivial solution, let us look for a solution $w$ such that

$$w \to \kappa, \quad \text{as} \quad s \to \infty.$$
Inner expansion

We write

\[ w = \kappa + \bar{w}, \]

and look for \( \bar{w} \) such that

\[ \bar{w} \to 0 \text{ as } s \to \infty. \]

The equation to be satisfied by \( \bar{w} \) is the following:

\[ \partial_s \bar{w} = \mathcal{L} \bar{w} + \bar{B}(\bar{w}) + \mu |\nabla \bar{w}|^{q_c}, \]

where \( q_c = \frac{2p}{p+1} \),

\[ \mathcal{L} v = \partial_y^2 v - \frac{1}{2} y \partial_y v + v, \]

and

\[ \bar{B}(\bar{w}) = |\bar{w} + \kappa|^{p-1}(\bar{w} + \kappa) - \kappa^p - p\kappa^{p-1}\bar{w}. \]

Note that \( \bar{B} \) is quadratic:

\[ \left| \bar{B}(\bar{w}) - \frac{p}{2\kappa} \bar{w}^2 \right| \leq C|\bar{w}^3|. \]
The linear operator

Note that $\mathcal{L}$ is self-adjoint in $D(\mathcal{L}) \subset L^2_\rho(\mathbb{R})$ where

$$L^2_\rho(\mathbb{R}) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}) \mid \int_{\mathbb{R}} (f(y))^2 \rho(y) dy < \infty \right\}$$

and

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{\sqrt{4\pi}}.$$

The spectrum of $\mathcal{L}$ is explicitly given by

$$\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \mid m \in \mathbb{N} \right\}.$$

All the eigenvalues are simple, and the eigenfunctions $h_m$ are (rescaled) Hermite polynomials, with

$$\mathcal{L}h_m = \left( 1 - \frac{m}{2} \right)h_m.$$

In particular, for $\lambda = 1, \frac{1}{2}, 0$, the eigenfunctions are $h_0(y) = 1$, $h_1(y) = y$ and $h_2(y) = y^2 - 2$. 
Naturally, we expand $\overline{w}(y, s)$ according to the eigenfunctions of $\mathcal{L}$:

$$\overline{w}(y, s) = \sum_{m=0}^{\infty} \overline{w}_m(s) h_m(y).$$

Since $h_m$ for $m \geq 3$ correspond to negative eigenvalues of $\mathcal{L}$, assuming $\overline{w}$ even in $y$, we may consider that

$$\overline{w}(y, s) = \overline{w}_0(s) h_0(y) + \overline{w}_2(s) h_2(y),$$

with

$$\overline{w}_0, \overline{w}_2 \to 0 \text{ as } s \to \infty.$$ 

Plugging this in the equation to be satisfied by $\overline{w}$:

$$\partial_s \overline{w} = \mathcal{L} \overline{w} + \overline{B}(\overline{w}) + \mu |\partial_y \overline{w}|^{q_c},$$
we first see that
\[ \mu |\partial_y w|^q_c = \mu 2^q_c |y|^q_c |w_2|^q_c, \]
then, projecting on \( h_0 \) and \( h_2 \), we get the following ODE system:
\[ \begin{align*}
\bar{w}_0' &= \bar{w}_0 + \frac{p}{2\kappa} (\bar{w}_0^2 + 8\bar{w}_2^2) + \tilde{c}_0 |\bar{w}_2|^q_c + O \left( |\bar{w}_0|^3 + |\bar{w}_2|^3 \right), \\
\bar{w}_2' &= 0 + \frac{p}{\kappa} (\bar{w}_0 \bar{w}_2 + 4\bar{w}_2^2) + \tilde{c}_2 |\bar{w}_2|^q_c + O \left( |\bar{w}_0|^3 + |\bar{w}_2|^3 \right),
\end{align*} \]
where
\[ 1 < q_c = \frac{2p}{p + 1} < 2, \quad \tilde{c}_0 = \mu 2^q_c \left( \int_{\mathbb{R}} |y|^{q_c} \rho \right), \quad \tilde{c}_2 = \mu q_c 2^{q_c-2} \left( \int_{\mathbb{R}} |y|^{q_c} \rho \right). \]
Note that the sign of \( \tilde{c}_0 \) and \( \tilde{c}_2 \) is the same as for \( \mu \).
Looking at the equation to be satisfied by $\bar{w}_2$

Let us write it as follows:

\[
\bar{w}_2' = \tilde{c}_2 |\bar{w}_2|^{q_c} \left(1 + O\left(|\bar{w}_2|^{2-q_c}\right)\right) + \frac{p}{\kappa} (\bar{w}_0 \bar{w}_2) + O\left(|\bar{w}_0|^3\right).
\]

Assuming that

\[
|\bar{w}_0 \bar{w}_2| \ll |\bar{w}_2|^{q_c}, \quad |\bar{w}_0|^3 \ll |\bar{w}_2|^{q_c}, \quad \text{(H1)}
\]

we end-up with

\[
\bar{w}_2' \sim \text{sign}(\mu) |\tilde{c}_2| |\bar{w}_2|^{q_c},
\]

which yields

\[
\bar{w}_2 = -\text{sign}(\mu) \frac{B}{s^{\frac{1}{q_c-1}}}, \quad \text{for some } B > 0.
\]

(remember that $q_c = \frac{2p}{p+1} \in (1, 2)$).
Looking at the equation to be satisfied by $\bar{w}_0$

Let us write it as follows:

$$w'_0 = \bar{w}_0 (1 + O(\bar{w}_0)) + \tilde{c}_0 |\bar{w}_2|^{qc} (1 + O(|\bar{w}_2|^{2-qc})) .$$

Assuming that

$$|w'_0| \ll \bar{w}_0, \quad |w'_0| \ll |\bar{w}_2|^{qc}, \quad (H2)$$

we end-up with

$$\bar{w}_0 \sim -\tilde{c}_0 |\bar{w}_2|^{qc} \sim -\frac{\tilde{c}_0 B^{qc}}{q_{c-1}} \ll \bar{w}_2 = -\text{sign}(\mu) \frac{B}{s^{q_c-1}} .$$

Note that both hypotheses (H1) and (H2) are satisfied by the found solution.
Conclusion for the inner expansion

Recalling the ansatz

\[ w(y, s) = \kappa + \bar{w}(y, s) = \kappa + \bar{w}_0(s)h_0(y) + \bar{w}_2(s)h_2(y) \]

with \( h_2(y) = y^2 - 2 \),

we end-up with

\[ w(y, s) = \kappa - \text{sign}(\mu)B \frac{y^2}{s^{2\beta}} + 2\text{sign}(\mu)B \frac{1}{s^{2\beta}} + o \left( \frac{1}{s^{2\beta}} \right), \]

with

\[ \beta = \frac{1}{2(q_c - 1)} = \frac{p + 1}{2(p - 1)} > \frac{1}{2}. \]

Remark: This expansion is valid in \( L^2_\rho \) and uniformly on compact sets by parabolic regularity. However, for \( y \) bounded, we see no shape: the expansion is asymptotically a constant.

Idea: What if \( z = \frac{y}{s^\beta} \) is the relevant space variable for the solution shape?
Outer expansion

To have a *shape*, following the inner expansion, *(valid for $|y|$ bounded)*,

$$ w(y, s) = \kappa - \text{sign}(\mu)Bz^2 + 2\text{sign}(\mu)B \frac{1}{s^{2\beta}} + o \left( \frac{1}{s^{2\beta}} \right) \quad \text{with} \quad z = \frac{y}{s^\beta}, $$

let us look for a solution of the following form *(valid for $|z|$ bounded):*

$$ w(y, s) = \bar{f}_\mu(z) + \frac{a}{s^{2\beta}} + O\left( \frac{1}{s^\nu} \right), \quad \nu > 2\beta, $$

with $z = \frac{y}{s^\beta}, \bar{f}_\mu(0) = \kappa$ and $\bar{f}_\mu$ bounded.

Plugging this ansatz in the equation,

$$ \partial_s w = \partial_y^2 w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^q, $$

then, keeping only the main order, we get

$$ -\frac{1}{2} z \bar{f}'_\mu(z) - \frac{1}{p-1} \bar{f}_\mu(z) + (\bar{f}_\mu(z))^p = 0, $$

hence, $\bar{f}_\mu(z) = \left( p - 1 + \bar{b}_\mu |z|^2 \right)^{-\frac{1}{p-1}},$ for some constant $\bar{b}_\mu > 0.$
Matching asymptotics

For $y$ bounded, both the inner expansion (valid for $|y|$ bounded)

$$w(y, s) = \kappa - \text{sign}(\mu)B\frac{y^2}{s^{2\beta}} + 2\text{sign}(\mu)B\frac{1}{s^{2\beta}} + o\left(\frac{1}{s^{2\beta}}\right),$$

and the outer expansion (valid for $|z|$ bounded)

$$w(y, s) = \bar{f}_\mu(z) + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right) \text{ where } \nu > 2\beta, \ z = \frac{y}{s^\beta}$$

and

$$\bar{f}_\mu(z) = \left(p - 1 + \bar{b}_\mu|z|^2\right)^{-\frac{1}{p-1}} = \kappa - \frac{\bar{b}_\mu\kappa}{(p - 1)^2}z^2 + O(z^4),$$

have to agree. Therefore,

$$\text{sign}(\mu)B = \frac{\bar{b}_\mu\kappa}{(p - 1)^2} \text{ and } a = 2\text{sign}(\mu)B.$$
Conclusion of the formal approach

We have just derived the blow-up profile for $|y| \leq Ks^\beta$:

$$\varphi(y, s) = \bar{f}_\mu \left( \frac{y}{s^\beta} \right) + \frac{a}{s^{2\beta}} = \left( p - 1 + \bar{b}_\mu \frac{|y|^2}{s^{2\beta}} \right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}},$$

where

$$\beta = \frac{p + 1}{2(p - 1)}, \quad \bar{b}_\mu = \frac{(p - 1)^2}{\kappa} B \text{ and } a = 2B,$$

with

$$B = \left[ 2^{q_c-2} q_c (q_c - 1) \int_{\mathbb{R}} |y|^{q_c} \rho \right]^{-\frac{1}{q_c - 1}} \mu^{-\frac{p+1}{p-1}}.$$
Strategy of the proof

We follow the strategy used by Bricmont and Kupiainen (1994) then Merle and Z. (1997) for the standard semilinear heat equation, based on:

- The reduction of the problem to a finite-dimensional one;
- The solution of the finite-dimensional problem thanks to the degree theory.

This strategy was later adapted for:

- the present equation with subcritical gradient exponent $q < q_c$ in Ebde and Z. (2011);
- the Ginzburg-Landau equation:

$$\partial_t u = (1 + i\beta) \Delta u + (1 + i\delta)|u|^{p-1}u - \gamma u$$

in Z. (1998) and Masmoudi and Z. (2008);

- the supercritical gKdV and NLS in Côte, Martel and Merle (2011);

- the semilinear wave equation

$$\partial_t^2 u = \partial_x^2 u + |u|^{p-1}u$$

in Côte and Z. (2013), for the construction of a blow-up solution showing multi-solitons.
Construction of solutions of PDEs with prescribed behavior

More generally, we are in the framework of constructing a solution to some PDE with some \textit{prescribed behavior}:

- NLS: Merle (1990), Martel and Merle (2006);
- water waves: Ming-Rousset-Tzvetkov (2013),
- Schrödinger maps: Merle-Raphaël-Rodniansky (2013),
- etc....
The strategy of the proof ($N = 1$)

We recall our aim: to construct a solution $w(y, s)$ of the equation in similarity variables:

$$\partial_s w = \partial^2_y w - \frac{1}{2} y \partial_y w - \frac{w}{p-1} + |w|^{p-1} w + \mu |\partial_y w|^{q_c},$$

such that

$$w(y, s) \sim \varphi(y, s) \quad \text{where} \quad \varphi(y, s) = \left( p - 1 + \bar{b} \mu \frac{|y|^2}{s^{2\beta}} \right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}}.$$

**Idea:** We linearize around $\varphi(y, s)$ by introducing

$$v(y, s) = w(y, s) - \varphi(y, s).$$
In that case, our aim becomes to construct \( v(y, s) \) such that

\[
\|v(s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty,
\]

and \( v(y, s) \) satisfies for all \( s \geq s_0 \) and \( y \in \mathbb{R} \),

\[
\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s),
\]

where

\[
\mathcal{L}v = \partial^2_y v - \frac{1}{2}y\partial_y v + v,
\]

\[
V(y, s) = p \varphi(y, s)^{p-1} - \frac{p}{p-1},
\]

\[
B(v) = |\varphi + v|^{p-1}(\varphi + v) - \varphi^p - p\varphi^{p-1}v,
\]

\[
G(\partial_y v) = \mu|\partial_y \varphi + \partial_y v|^{q_c} - \mu|\partial_y \varphi|^{q_c},
\]

\[
R(y, s) = -\partial_s \varphi + \partial^2_y \varphi - \frac{1}{2}y\partial_y \varphi - \frac{\varphi}{p-1} + \varphi^p + \mu|\partial_y \varphi|^{q_c}.
\]
Effect of the different terms

- **The linear term**: Its spectrum is given by \( \{1 - \frac{m}{2}, \mid m \in \mathbb{N}\} \) and its eigenfunctions are Hermite polynomials with \( \mathcal{L}h_m = (1 - \frac{m}{2})h_m \).

  Note that we have two positive directions \( \lambda = 1, \frac{1}{2} \) and a null direction \( \lambda = 0 \).

- **The potential term** \( V \): it has two fundamental properties:
  
  (i) \( V(., s) \to 0 \) in \( L^2_p(\mathbb{R}) \) as \( s \to \infty \). In practice, the effect of \( V \) in the blow-up area \( (|y| \leq Ks^\beta) \) is regarded as a perturbation of the effect of \( \mathcal{L} \) (except on the null mode).

  (ii) \( V(., s) \to -\frac{p}{p-1} \) as \( s \to \infty \) and \( \frac{|y|}{s^\beta} \to \infty \). Since \( -\frac{p}{p-1} < -1 \) and 1 is the largest eigenvalue of the operator \( \mathcal{L} \), outside the blow-up area (i.e. for \( |y| \geq Ks^\beta \)), we may consider that the operator \( \mathcal{L} + V \) has negative spectrum, hence, easily controlled.

- **The nonlinear term in** \( v \): It is quadratic: \( |B(v)| \leq C|v|^2 \),

- **The nonlinear term in** \( \partial_y v \): It is sublinear: \( \|G(\partial_y v)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{s}}\|\partial_y v\|_{L^\infty(\mathbb{R})} \).

- **The rest term**: It is small: \( \|R(., s)\|_{L^\infty} \leq \frac{C}{s} \).
From the properties of the profile and the potential, the variable

\[ z = \frac{y}{s^{\beta}} \]

plays a fundamental role, and our analysis will be different in the regions

\[ |z| > K \text{ and } |z| < 2K. \]

The linear operator will be predominant on all the modes, except on the null mode (i.e. with the eigenfunction \( h_2(y) \)) where the terms \( Vv \) and \( G(\partial_y v) \) will play a crucial role.
**Decomposition of $v(y, s)$ into “”inner” and “outer” parts**

Consider a cut-off function

$$
\chi(y, s) = \chi_0 \left( \frac{|y|}{K s^\beta} \right),
$$

where $\chi_0 \in C^\infty([0, \infty), [0, 1])$, s.t. $\text{supp}(\chi_0) \subset [0, 2]$ and $\chi_0 \equiv 1$ in $[0, 1]$. Then, introduce

$$
v(y, s) = v_{\text{inner}}(y, s) + v_{\text{outer}}(y, s),
$$

with

$$
v_{\text{inner}}(y, s) = v(y, s)\chi(y, s) \quad \text{and} \quad v_{\text{outer}}(y, s) = v(y, s)(1 - \chi(y, s)).
$$

Note that

$$
\text{supp } v_{\text{inner}}(s) \subset B(0, 2Ks^\beta), \quad \text{supp } v_{\text{outer}}(s) \subset \mathbb{R}^N \setminus B(0, Ks^\beta).
$$

**Remark:** $v_{\text{outer}}(y, s)$ is easily controlled, because $\mathcal{L} + V$ has a negative spectrum (less than $1 - \frac{p}{p-1} + \epsilon < 0$).
Decomposition of the “inner” part

We decompose $v_{\text{inner}}$, according to the sign of the eigenvalues of $\mathcal{L}$:

$$v_{\text{inner}}(y, s) = \sum_{m=0}^{2} v_m(s) h_m(y) + v_-(y, s),$$

where $v_m$ is the projection of $v_{\text{inner}}$ (and not $v$ on $h_m$, and $v_-(y, s) = P_-(v_{\text{inner}})$ with $P_-$ being the projection on the negative subspace $E_- \equiv \text{Span}\{h_m \mid m \geq 3\}$ of the operator $\mathcal{L}$.

**Remark:** $v_-(y, s)$ is easily controlled because the spectrum of $\mathcal{L}$ restricted to $E_-$ is less than $-\frac{1}{2}$.

It remains then to control $v_0$, $v_1$ and $v_2$. 
Control of $\nu_2$

This is delicate, because it corresponds to the direction $h_2(y)$, the null mode of the linear operator $\mathcal{L}$.

Projecting the equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s)$$

on $h_2(y)$, and recalling that $\mathcal{L}h_2 = 0$, we need to refine the contributions of $Vv$ and $G(\partial_y v)$ to the linear term (this is delicate), and write:

$$v'(s) = -\frac{2}{s}v_2(s) + O\left(\frac{1}{s^{4\beta}}\right) + O\left(\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right).$$

Working in the slow variable $\tau = \log s = \log |\log(T - t)|$, we see that

$$\frac{d}{d\tau} v_2 = -2v_2 + O\left(\frac{1}{s^{4\beta - 1}}\right) + O\left(s\|v(s)\|^2_{W^{1,\infty}(\mathbb{R})}\right),$$

which shows a negative eigenvalue.

**Conclusion:** $\nu_2$ can be controlled as well.

We are left only with two components $\nu_0$ and $\nu_1$: A finite dimensional problem.
Dealing with \( v_0 \) and \( v_1 \)

These remaining components correspond respectively to the projections along \( h_0(y) = 1 \) and \( h_1(y) = y \), the positive directions of \( \mathcal{L} \). Projecting the equation

\[
\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s)
\]
on \( h_m(y) \) with \( m = 0, 1 \), we write

\[
v'_0(s) = v_0(s) + O \left( \frac{1}{s^{2\beta+1}} \right) + O \left( \|v(s)\|_{W^{1,\infty}(\mathbb{R})}^2 \right),
\]
\[
v'_1(s) = \frac{1}{2} v_1(s) + O \left( \frac{1}{s^{2\beta+1}} \right) + O \left( \|v(s)\|_{W^{1,\infty}(\mathbb{R})}^2 \right).
\]

Since all the other components are easy to control, we may assume that

\[
v(y, s) = v_0(s)h_0(y) + v_1(s)h_1(s) = v_0(s) + v_1(s)y,
\]
ending with a “baby” problem, \textit{which is two-dimensional}, with initial data at \( s = s_0 \) given by

\[
v_0(s_0) = d_0, \quad v_1(s_0) = d_1,
\]
Solution of the baby problem

Recall the baby problem:

\[ v'_0(s) = v_0(s) + O \left( \frac{1}{s^{2\beta+1}} \right) + O \left( v_0(s)^2 \right) + O \left( v_1(s)^2 \right), \]

\[ v'_1(s) = \frac{1}{2} v_1(s) + O \left( \frac{1}{s^{2\beta+1}} \right) + O \left( v_0(s)^2 \right) + O \left( v_1(s)^2 \right), \]

with initial data at \( s = s_0 \) given by

\[ v_0(s_0) = d_0, \quad v_1(s_0) = d_1. \]

This problem can be easily solved by contradiction, based on Index Theory:

There exist a particular value \((d_0, d_1) \in \mathbb{R}^2\) such that the “baby” problem has a solution \((v_0(s), v_1(s))\) which converges to \((0, 0)\) as \( s \to \infty \).
Conclusion for the full problem

For the full problem (which is *infinite-dimensional*), recalling that

\[ v(y, s) = v_{\text{inner}}(y, s) + v_{\text{outer}}(y, s) = \sum_{m=0}^{2} v_m(s) h_m(y) + v_-(y, s) + v_{\text{outer}}(y, s), \]

and that all the three other components correspond to *negative* eigenvalues, hence easily converging to zero, we have the following statement:

*Consider the equation*

\[ \partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s) \]

*equipped with initial data at* \( s = s_0 \):

\[ \psi_{s_0, d_0, d_1}(y) = \left( d_0 h_0(y) + d_1 h_1(y) \right) \chi(2y, s_0). \]

*Then, there exists a particular value* \( (d_0, d_1) \) *such that the corresponding solution* \( v(y, s) \) *exists for all* \( s \geq s_0 \) *and* \( y \in \mathbb{R} \), *and satisfies*

\[ \|v(y, s)\|_{L^\infty(\mathbb{R})} \to 0 \text{ as } s \to \infty. \]
End of the proof of the existence proof

Introducing

\[ T = e^{-s_0}, \]

and recalling that

\[ v(y, s) = w(y, s) - \phi(y, s) \quad \text{and} \quad u(x, t) = (T - t)^{-\frac{1}{p-1}} w\left( \frac{x}{\sqrt{T-t}}, -\log(T-t) \right), \]

and

\[ \phi(y, s) = \overline{f}_\mu \left( \frac{y}{s^\beta} \right) + \frac{a}{s^{2\beta}}, \]

we derive the existence of \( u(x, t) \), a solution to the equation

\[ \partial_t u = \Delta u + \mu |\nabla u|^{q_c} + |u|^{p-1} u, \]

such that

\[ (T - t)^{-\frac{1}{p-1}} u(z\sqrt{T-t} \log(T-t))^{\frac{p+1}{2(p-1)}}, t) \sim \overline{f}_\mu(z) \text{ as } t \to T. \]

Using refined parabolic regularity estimates, we derive that:

- \( u(x, t) \) blows up only at the origin;

- the final profile satisfies \( u(x, T) \sim \left( \frac{2}{b_\mu} |x|^{-2} |\log |x||^{\frac{p+1}{p-1}} \right)^{\frac{1}{p-1}} \) as \( x \to 0 \).
Proof of the stability result ($N = 1$)

Let us recall the statement:

**Theorem (Tayachi and Z.: Stability)**

The constructed solution is stable with respect to perturbations in initial data in $W^{1,\infty}(\mathbb{R}^N)$.

**Proof**: It follows from the existence proof, through the interpretation of the parameters of the finite-dimensional problem in terms of the blow-up time and the blow-up point.

Consider $\hat{u}(x,t)$ the constructed solution, with initial data $\hat{u}_0$, blowing up at time $\hat{T}$ only at one blow-up point $\hat{a}$ (not necessarily 0).

Consider now $u_0 \in W^{1,\infty}(\mathbb{R})$ such that

$$u_0 = \hat{u}_0 + \epsilon_0 \quad \text{with} \quad \|\epsilon_0\|_{W^{1,\infty}(\mathbb{R})} \text{ small},$$

and $u(x,t)$ the corresponding maximal solution and $T(u_0) \leq +\infty$ its maximal existence time.

We would like to prove that $T(u_0) < +\infty$, and that $u(x,t)$ blows up only at one single point $a(u_0)$ with the same profile as for $\hat{u}(x,t)$, with

$$T(u_0) \to \hat{T} \quad \text{and} \quad a(u_0) \to \hat{a} \quad \text{as} \quad u_0 \to \hat{u}_0.$$
Our finite dimensional parameters

At this stage, we don’t even know that $T(u_0) < +\infty$, don’t mention the blow-up point $a(u_0)$.

Since our goal is to show that $T(u_0)$ and $a(u_0)$ are close to $\hat{T}$ and $\hat{a}$ respectively, let us study ALL the similarity variables versions of $u(x,t)$ considered with arbitrary $(T,a)$ close to $(\hat{T},\hat{a})$:

$$w_{u_0,T,a}(y,s) = e^{-\frac{s}{p-1}}u(a + ye^{\frac{s}{2}}, T - e^{-s})$$

and

$$v_{u_0,T,a}(y,s) = w_{u_0,T,a}(y,s) - \varphi(y,s) = e^{-\frac{s}{p-1}}u(a + ye^{\frac{s}{2}}, T - e^{-s}) - \varphi(y,s),$$

where the profile:

$$\varphi(y,s) = \bar{f}_\mu\left(\frac{y}{s^\beta}\right) + \frac{a}{s^{2\beta}}.$$
The stability problem as an “existence” problem

Note that *for any* \((T, a)\), \(v_{u_0, T, a}(y, s)\) satisfies the *same* equation as for the existence proof:

\[
\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s),
\]

with initial data, say at some time \(s = s_0\) large enough, given by

\[
v_{u_0, T, a}(y, s_0) = \bar{\psi}_{u_0, T, a}(y) \equiv e^{-\frac{s_0}{p-1}} u(a + ye^\frac{s_0}{2}, T - e^{-s_0}) - \varphi(y, s_0).
\]

**Idea:** These initial data depend on two parameters, exactly as in the existence proof, where initial data was

\[
\psi_{s_0, d_0, d_1}(y) = \left(d_0 h_0(y) + d_1 h_1(y)\right) \chi(2y, s_0),
\]

dependning also on two parameters.

It happens that the behaviors of

\[(d_0, d_1) \mapsto \psi_{s_0, d_0, d_1} \text{ and } (T, a) \mapsto \bar{\psi}_{u_0, T, a}\]

are similar, so the existence proof starting from \(\bar{\psi}_{u_0, T, a}\) works also, in the sense that:
Statement for the “new” existence problem

For $||u_0 - \hat{u}_0||_{W^{1,\infty}}$ small and $s_0$ large enough, there exists some $(\bar{T}(u_0), \bar{a}(u_0))$ such that the solution of equation

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(\partial_y v) + R(y, s),$$

with initial data at $s = s_0$ given by

$$\bar{\psi}_{u_0,T,a}(y) \equiv e^{-s_0 \frac{a}{p-1}} u(a + ye^{\frac{s_0}{2}}, T - e^{-s_0}) - \varphi(y, s_0)$$

with $(T, a) = (\bar{T}(u_0), \bar{a}(u_0))$, converges to 0 as $s \to \infty$.

But remember!

$$\bar{\psi}_{u_0,T(u_0),\bar{a}(u_0)}(y) = v_{u_0,T(u_0),\bar{a}(u_0)}(y, s_0),$$

so we know that solution: it is simply $v_{u_0,T(u_0),\bar{a}(u_0)}(y, s) !!!!
Therefore, this is in reality the statement we have just proved:

\[ \text{There exists some } (\bar{T}(u_0), \bar{a}(u_0)) \text{ such that } \|v_{u_0, \bar{T}(u_0), \bar{a}(u_0)}\|_{W^{1, \infty}} \to 0 \text{ as } s \to \infty. \]

Going back in the transformations, we see that for all \( t \in [0, \bar{T}(u_0)) \) and \( x \in \mathbb{R} \),

\[ u(x, t) = (\bar{T}(u_0) - t)^{-\frac{1}{p-1}} w_{u_0, \bar{T}(u_0), \bar{a}(u_0)}(y, s) = (\bar{T}(u_0) - t)^{-\frac{1}{p-1}} \left[ \varphi(y, s) + v_{u_0, \bar{T}(u_0), \bar{a}(u_0)}(y, s) \right] \]

where

\[ y = \frac{x - \bar{a}(u_0)}{\sqrt{\bar{T}(u_0) - t}} \quad \text{and} \quad s = -\log(\bar{T}(u_0) - t). \]

From this identity, we see that:

- The blow-up time of \( u(x, t) \) is in fact \( \bar{T}(u_0) \);
- \( u(x, t) \) blows up at the point \( \bar{a}(u_0) \);
- \( u(x, t) \) has \( \varphi(y, s) \) as blow-up profile, the same as for \( \hat{u}(x, t) \),

and this is the desired conclusion for the stability.
Thank you for your attention.