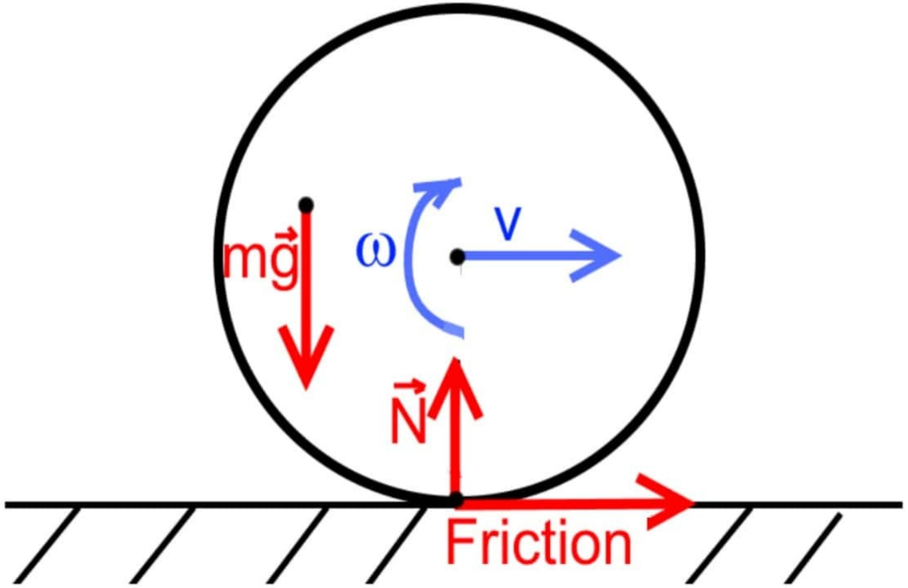
Systems with linear constraints on velocities



No sliding: $\omega + v\,r = 0$

$$\ddot{\varphi} + \sin\varphi = 0$$

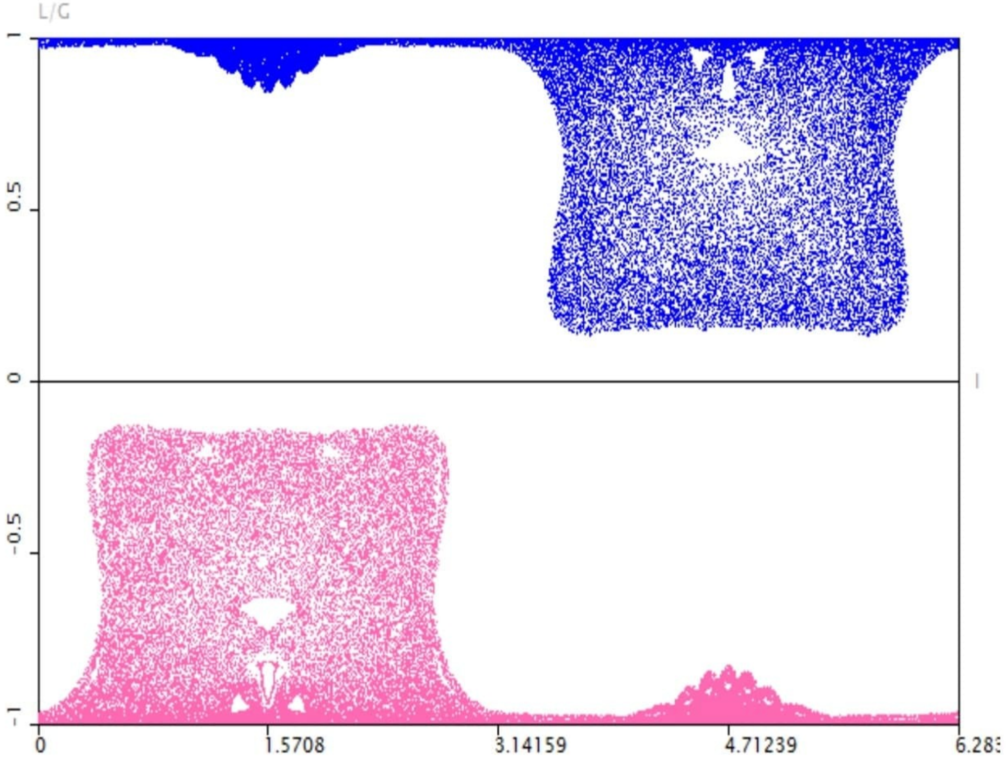
Friction does not perform work, so the energy is conserved

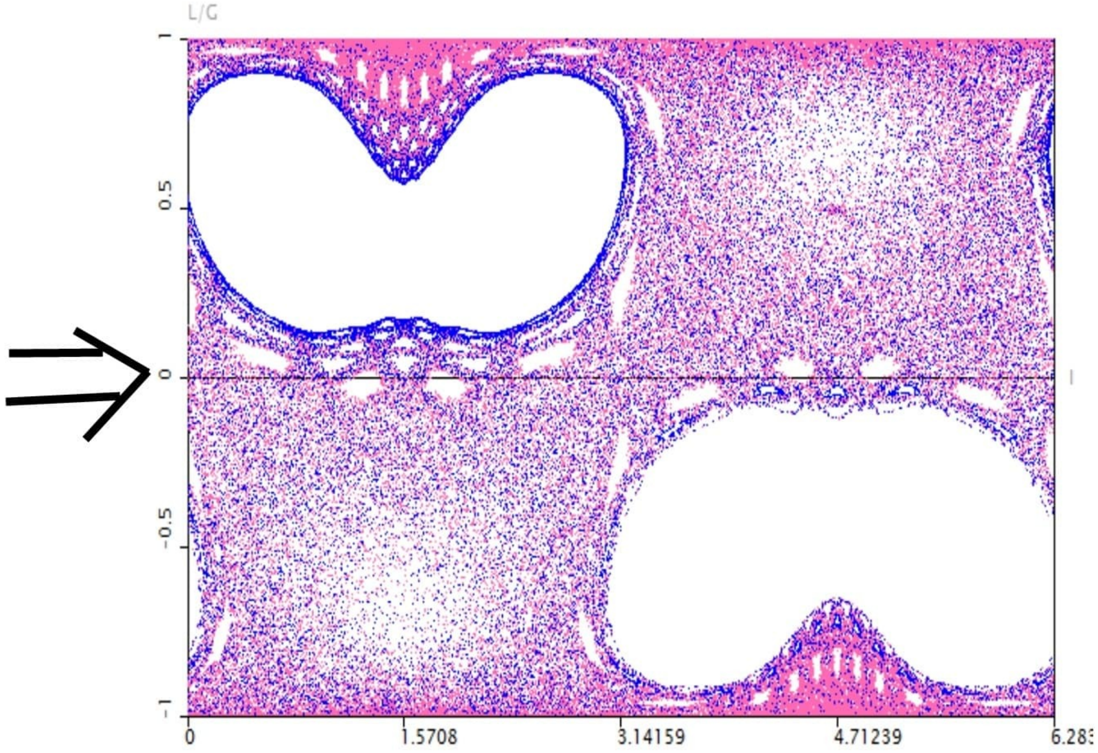
"Rubber ellipsoid": rolling without sliding or rotation

2 coordinates + 3 angles =
5 degrees of freedom =
10 variables
- 3 angular velocities
- 2 coordinates and 1 angle
(for the Euclidian symmetry)
= 4 variables remaining

Energy is conserved,
but the system on a 3-dimensional energy level
does not preserve the phase volume!

ATTRACTOR - REPELLER MERGER

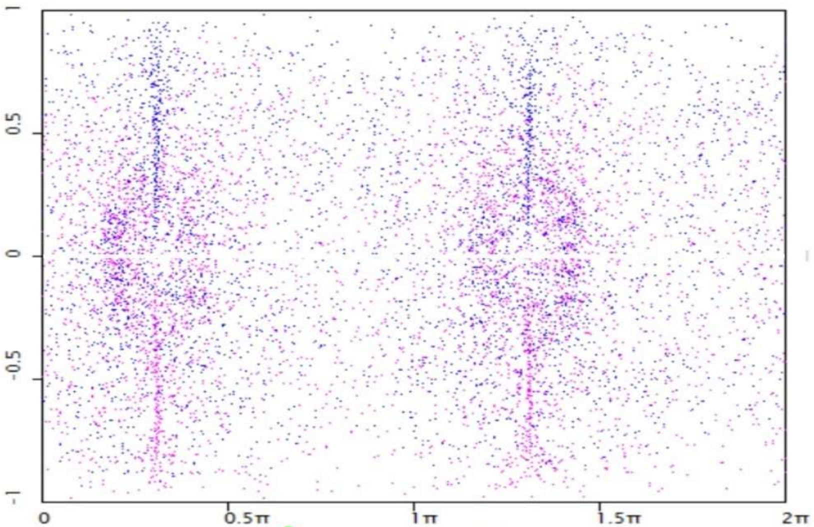




Chaplygin ball (rubber body) dynamics for different energies (by A.Kazakov)

THIRD TYPE OF CHAOS

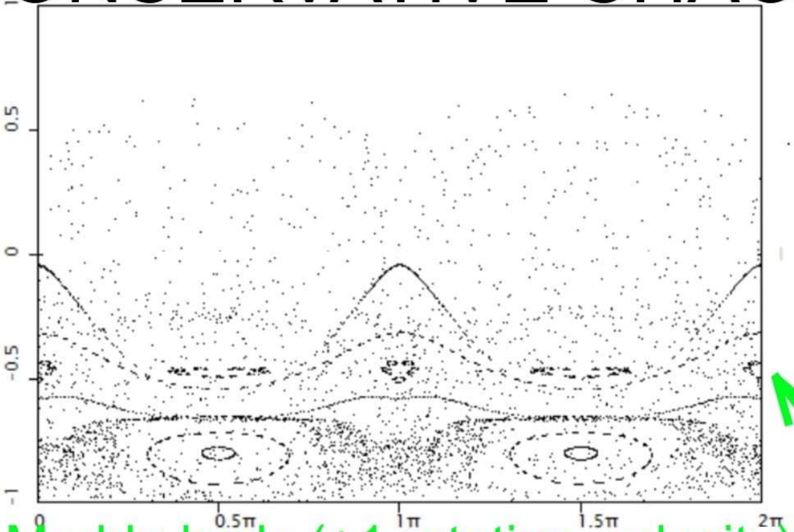
Classification of Chaos
(by Sergey Gonchenko)



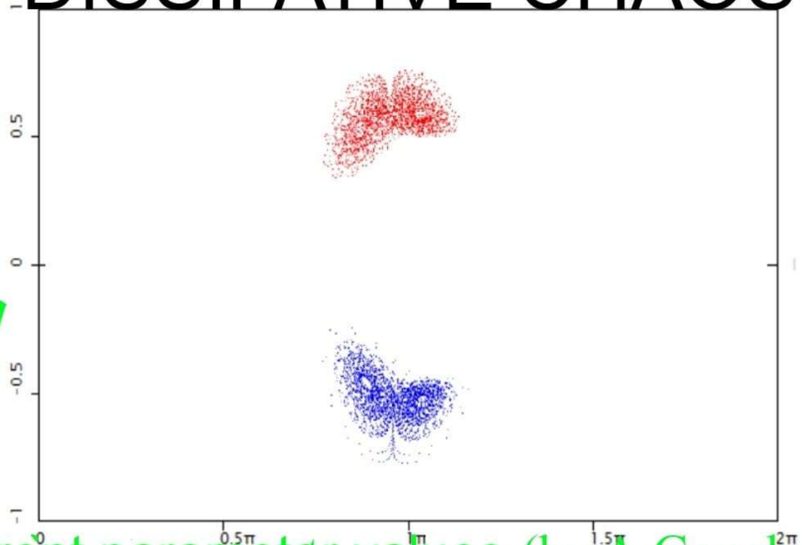
— attractor

— repeller

CONSERVATIVE CHAOS



DISSIPATIVE CHAOS

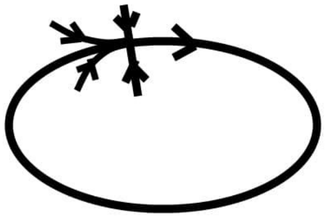


Marble body (+1 rotation velocity) dynamics for different parameter values (by A.Gonchenko)

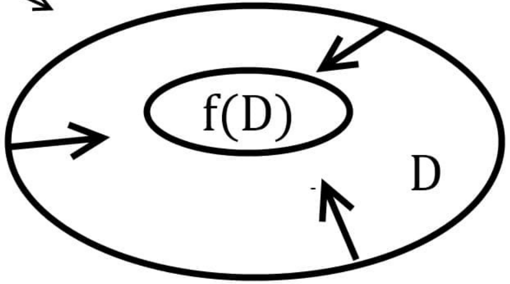
How to define attractor?

1. Attractor must attract something ?
2. Attractor must retain orbits
3. Attractor must be indecomposable
4. Attractors must exist !

An asymptotically stable transitive set does not always exist:



Conley-Ruelle-Hurley attractor: Chain-transitive set at the intersection of a sequence of nested absorbing domains

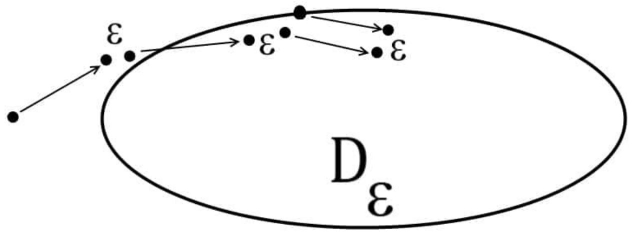


At least one CRH-attractor exists for every system!

The set of all points attainable from a given point by ε-orbits for any given fixed ε is an absorbing domain

Inside each absorbing domain lies a maximal attractor $A = \bigcap_{n>0} f^n(D_\varepsilon)$ $\quad f(cl(D_\varepsilon)) \subset D_\varepsilon$

If it is not chain-transitive, there is a smaller absorbing domain



...

The process stops at a CRH-attractor

For a conservative system with a compact phase space each point is chain-recurrent

1. Conservative chaos = the entire phase space is CRH-attractor

2. Dissipative chaos=a CRH-attractor that does not intersect CRH-repellers=

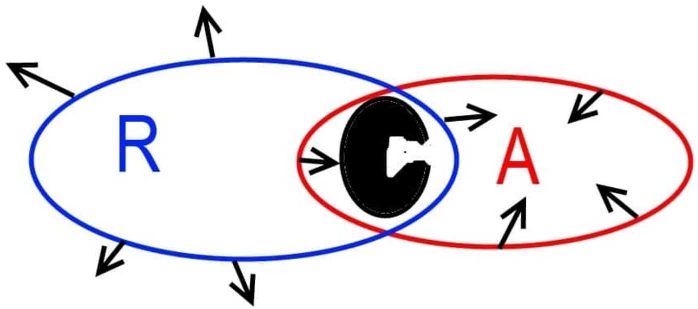
= it attracts some ε-orbits

3d type of chaos = Reversible Core =

= a CRH-attractor which is also a CRH-repeller =

= it does not attract anything

A reversible core is a limit of infinitely many attractors and repellers →



→ Numerically, RC looks like an intersection of an attractor and a repeller which almost coincide

Every elliptic point of a generic reversible 2Dmap is a reversible core

C^r

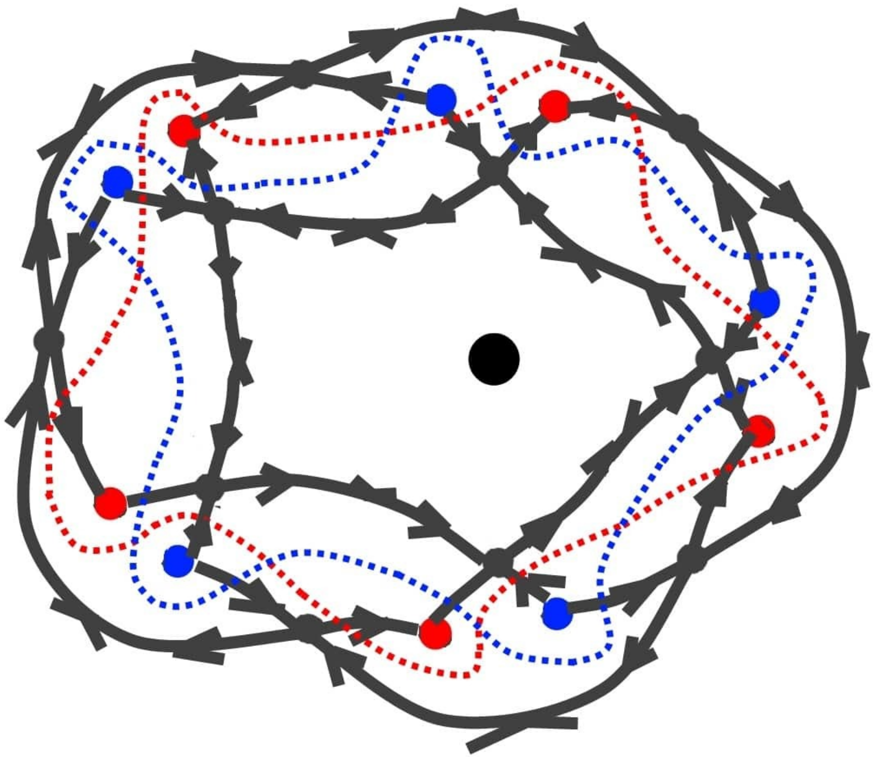
(due to non-conservative resonances)

$f^{-1} = g \circ f \circ g$

$g \circ g = id$

Symmetric fixed point:
g(x)=x,   f(x)=x
($x \in f(\mathrm{Fix}\ g) \cap \mathrm{Fix}\ g$)

$$\lambda_{1,2} = e^{\pm 2\pi i(\frac{p}{q} + \mu)}$$



← A=0

Resonance of order q :

$$\dot{z} = i\mu z + i \sum_{1 \le j \le \frac{q}{2}} \Psi_j |z|^{2j} z + iA(z^*)^{q-1} + iBz^{q+1} + iCz(z^*)^q$$

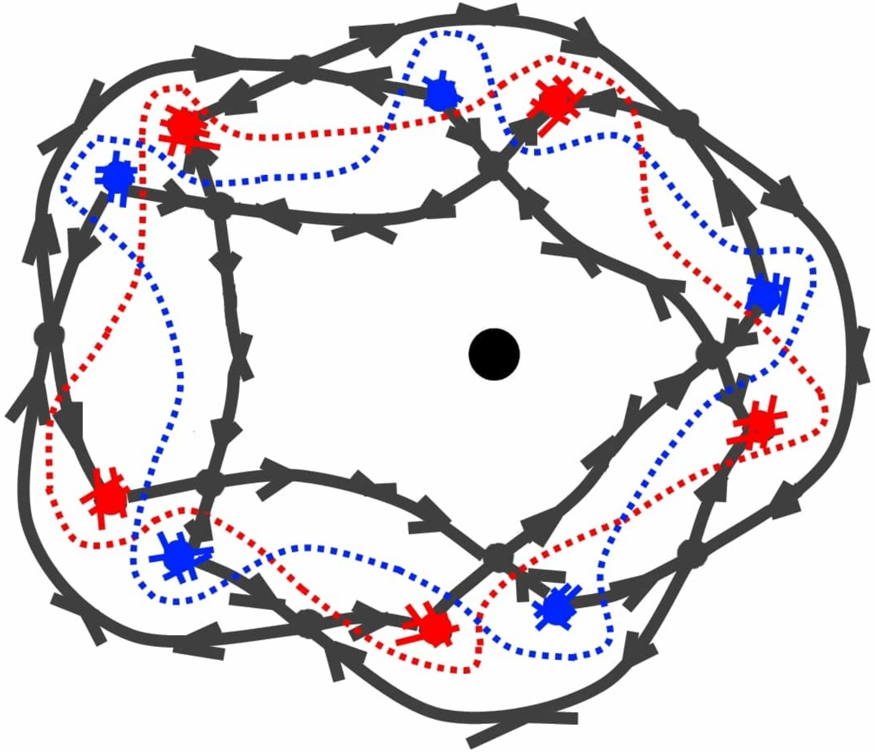
Gonchenko, Lamb, Rios, Turaev

Every elliptic point of a C^r-generic reversible map (with $\dim(\text{Fix } g) \geqslant n/2$)

is a limit of a sequence of hyperbolic attractors and repellers

(born from periodic spots)

Gonchenko,Turaev:



⇓

EVERYTHING can be born from periodic spots

⇓

A C^r-generic reversible map with an elliptic point is C^r-universal

Mechanical systems with nonholonomic constraints

Newton law: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = m_i \ddot{x}_i = F_i \qquad (i=1,\dots,N)$

$T = \sum_i m_i \frac{\dot{x}_i^2}{2}$

coordinate transformation $x \to q$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i'$$

Nonholonomic constraints

$$\sum_i a_{ji}(q)\dot{q}_i = 0 \qquad (j=1,\dots,s)$$

Conservative case: $F_i' = 0$

The energy $L - \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}$ is conserved

Holonomic constraint $G(q)=0 \longrightarrow G'(q)\dot{q}=0$

Symmetry

$$\frac{\partial L}{\partial q_j} = 0 \longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \longrightarrow \frac{\partial L}{\partial \dot{q}_j} = const$$

The energy is conserved

$$\sum_i \dot{q}_i F_i' = 0 \longrightarrow F_i' = \sum_j \mu_j a_{ji}$$

$$\frac{d}{dt}\sum_i a_{ji}(q)\dot{q}_i = 0 \qquad (j=1,\dots,s)$$

Mechanical systems with nonholonomic constraints

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = \sum_j \mu_j a_{ji}(q) \quad (i=1,\dots,N)$$

$$\sum_i a_{ji}(q)\dot q_i = 0 \quad (j=1,\dots,s)$$

μ_j are found from the conditions

$$\frac{d}{dt}\sum_i a_{ji}(q)\dot q_i = 0 \quad (j=1,\dots,s)$$

which make S of those equations redundant

$(2N-S)$ - dimensional system of ODEs for N coordinates and $(N-S)$ velocities

The energy $L(q,\dot q) - \sum_i \dot q_i \frac{\partial L}{\partial \dot q_i}$ is conserved

When the Lagrangian is quadratic in velocities the system is time-reversible

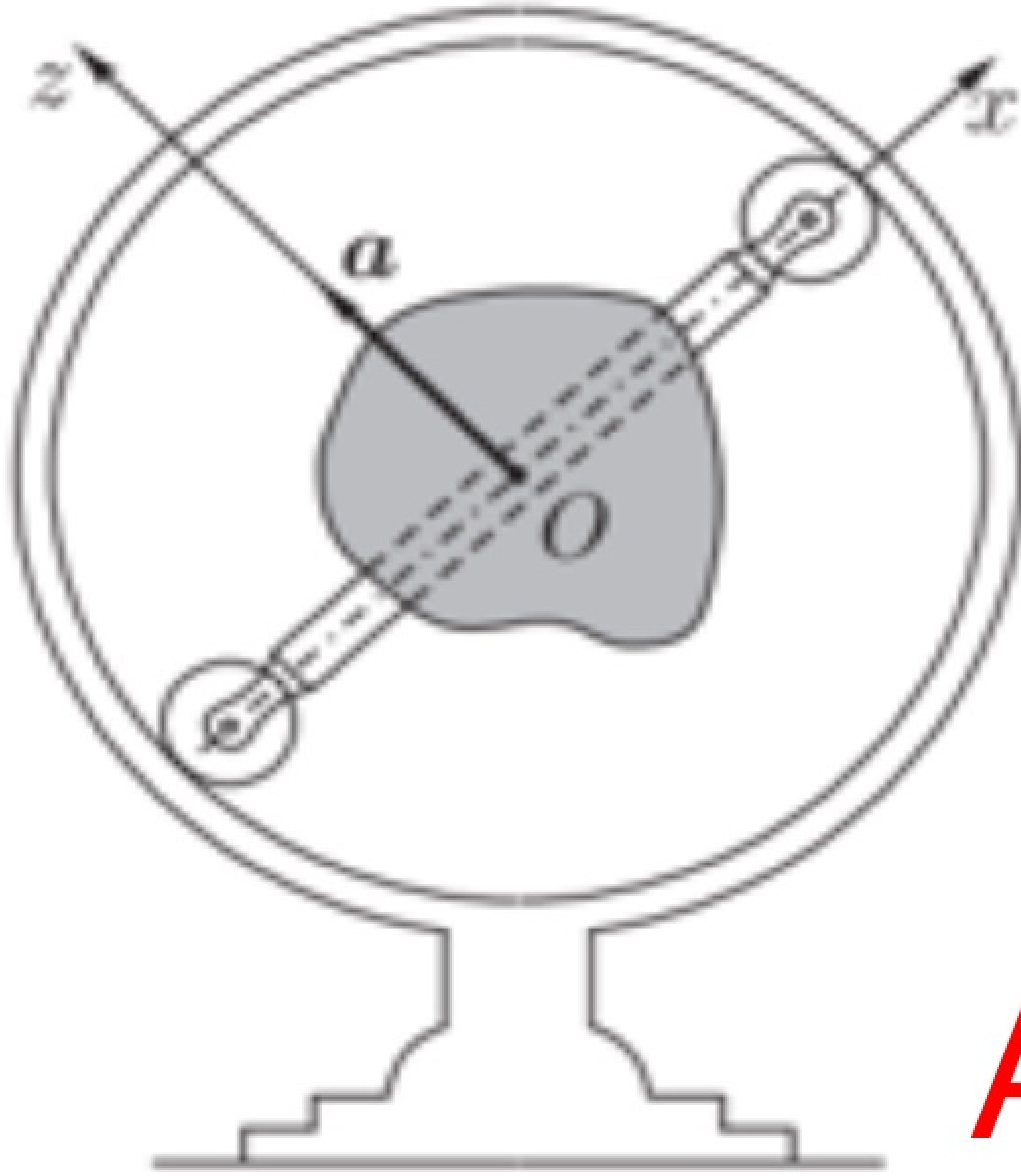
$Fix\, G = \{\dot q = 0\} = \{L(q,0) = E\}$ ⟶ $dim(Fix(G)) = N-1 \geqslant \frac{2N-S-2}{2}$ $\quad E < max\, L(q,0)$

⟶ $dim(Fix(G)) = -1$ $\quad E > max\, L(q,0)$

(symmetries may change the count)

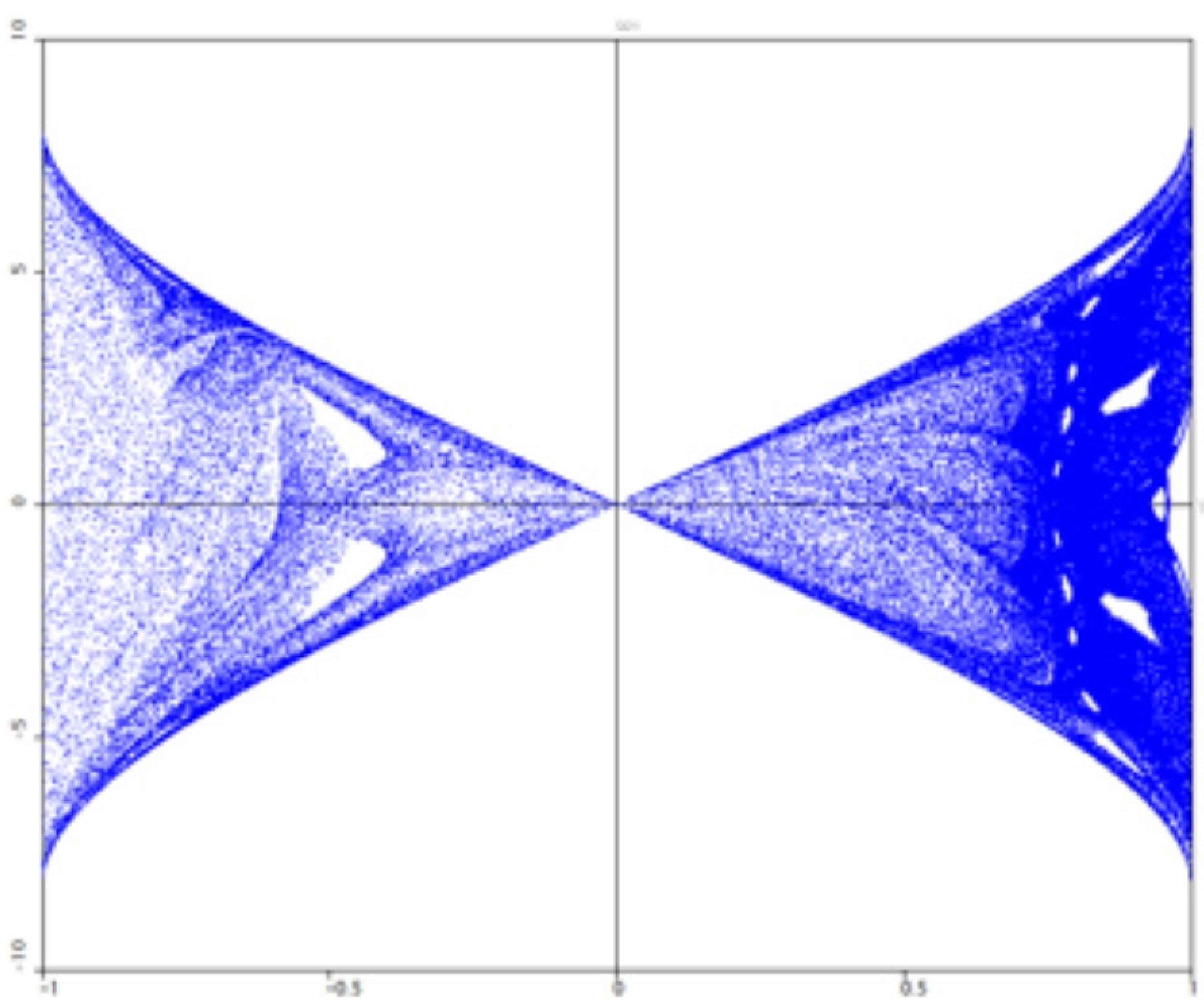
ATTRACTOR - REPELLER MERGER

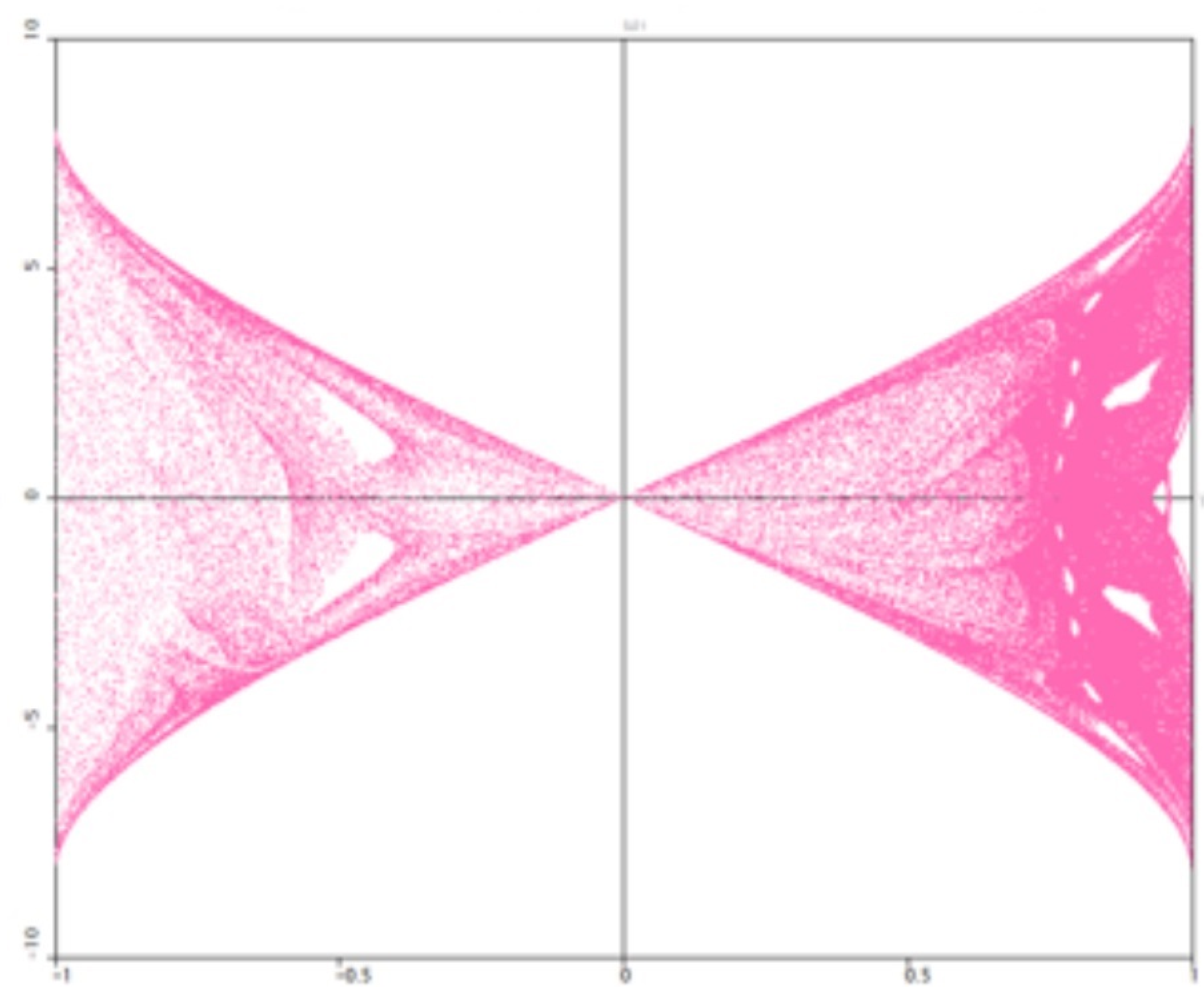
Chaplygin ball (rubber body) dynamics for different energies (by A.Kazakov)



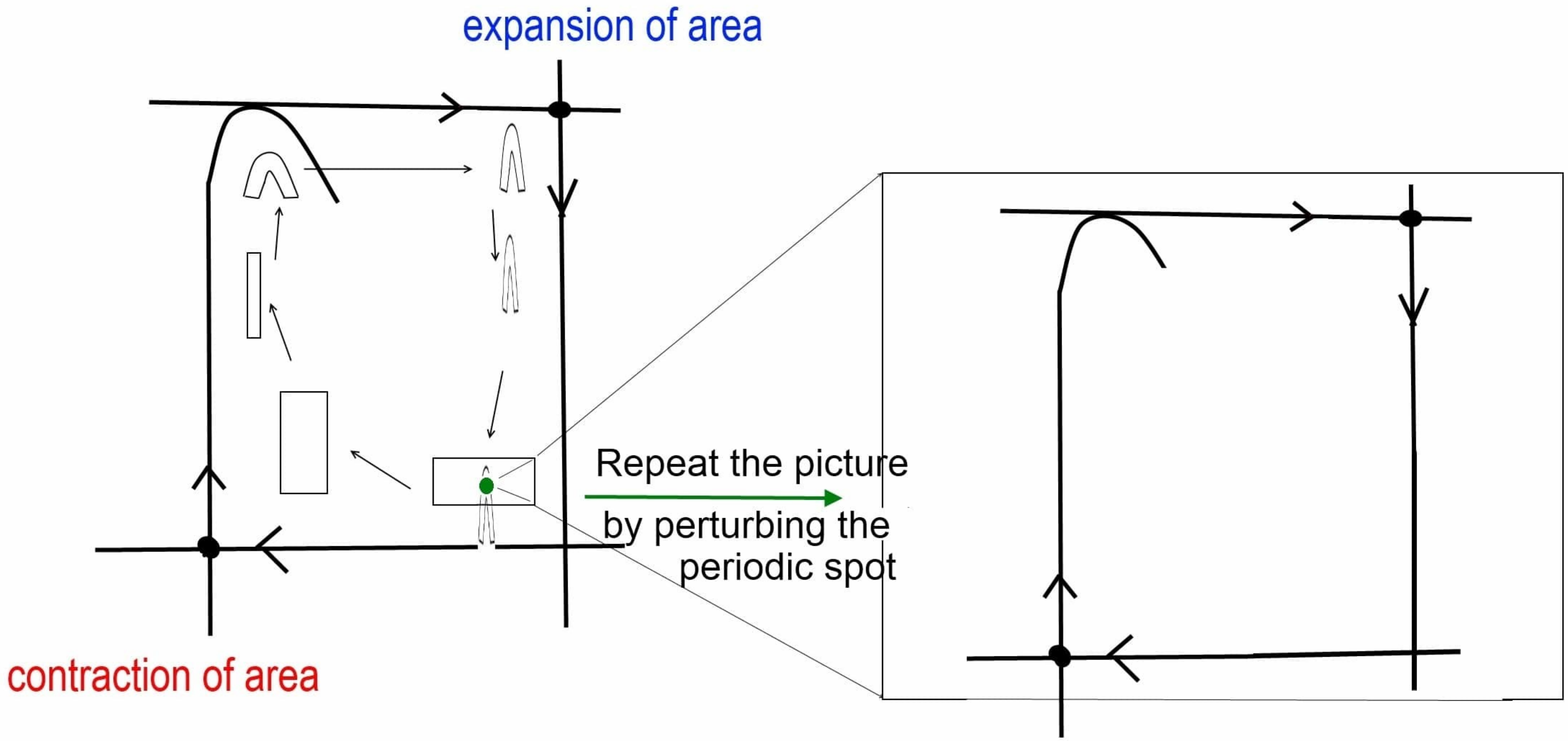
Suslov model

Attractor and Repeller by A.Kazakov

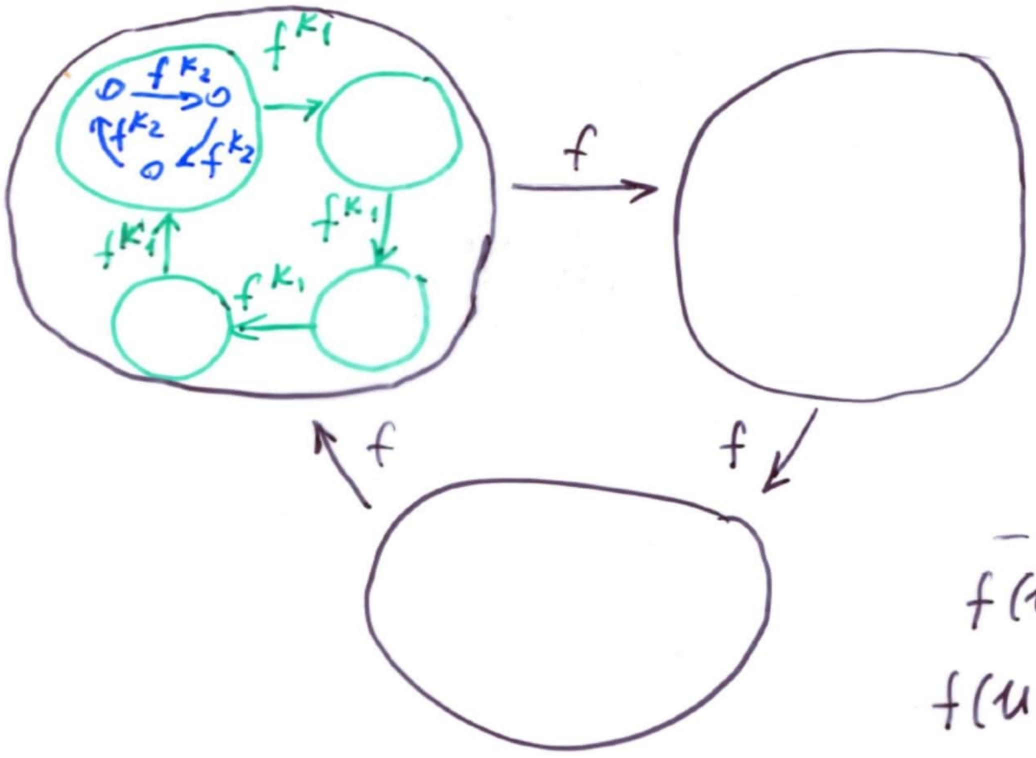




Heteroclinic cycles in non-dissipative (e.g. reversible) maps lead to solenoids



Solenoid



$$\mathcal{U}^1_1, \dots, \mathcal{U}^1_{k_1}$$
$$\cup$$
$$\mathcal{U}^2_1, \dots, \mathcal{U}^2_{k_2}$$
$$\cup$$
$$\mathcal{U}^3_1, \dots, \mathcal{U}^3_{k_3}$$
$$\cup$$

— — — — — —

$$f(\mathcal{U}^i_j) = \mathcal{U}^i_{j+1}, \quad j < k_i$$

$f(\mathcal{U}^i_{k_i}) \subset \mathrm{int}(\mathcal{U}^i_1)$ ⇓ $A = \bigcap_i \left(\bigcup_{j \le k_i} \mathcal{U}^i_j \right)$ attractor

$\mathcal{U}^i_1 \subset f(\mathrm{int}(\mathcal{U}^i_{k_i}))$ ⇓ $R = \bigcap_i \left(\bigcup_{j \le k_i} \mathcal{U}^i_j \right)$ repeller

even i | odd i → reversible core

$$\| Df^{k_i}|_{\mathcal{U}^i_1} \| \le 1 + \varepsilon_i$$
$$\| (Df^{k_i}|_{\mathcal{U}^i_1})^{-1} \| \le 1 + \varepsilon_i$$

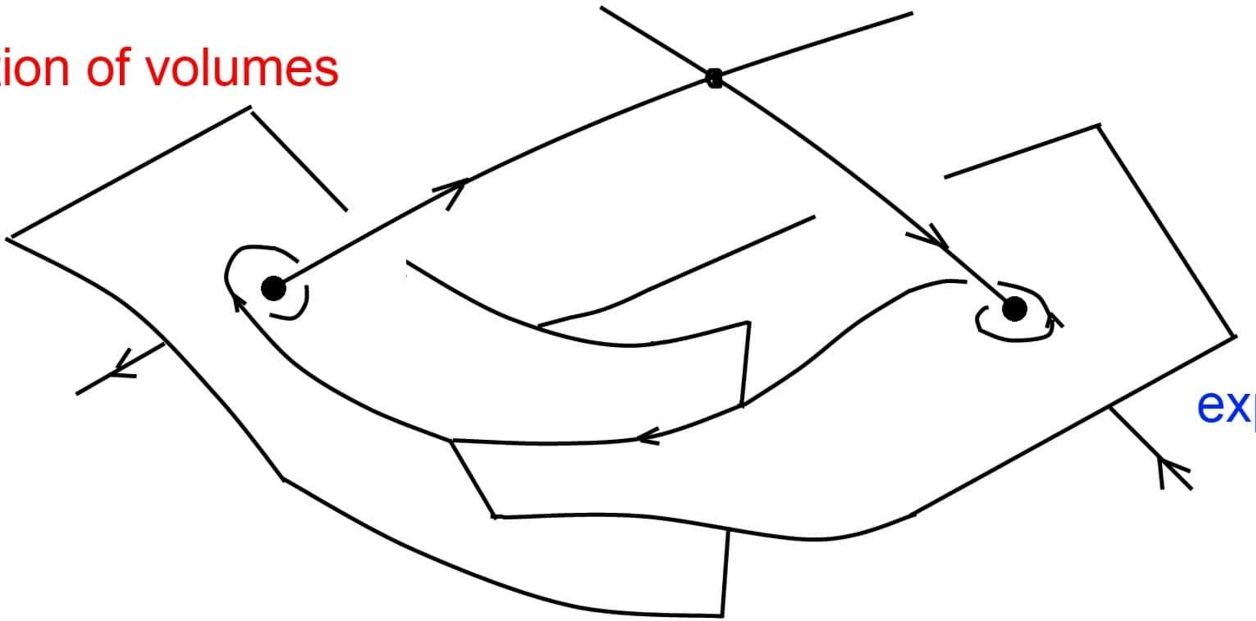
$\varepsilon_i \to 0$ ⇒ zero Lyapunov exponents

$i \to \infty$

Can Bonatti-Diaz blenders produce periodic spots?

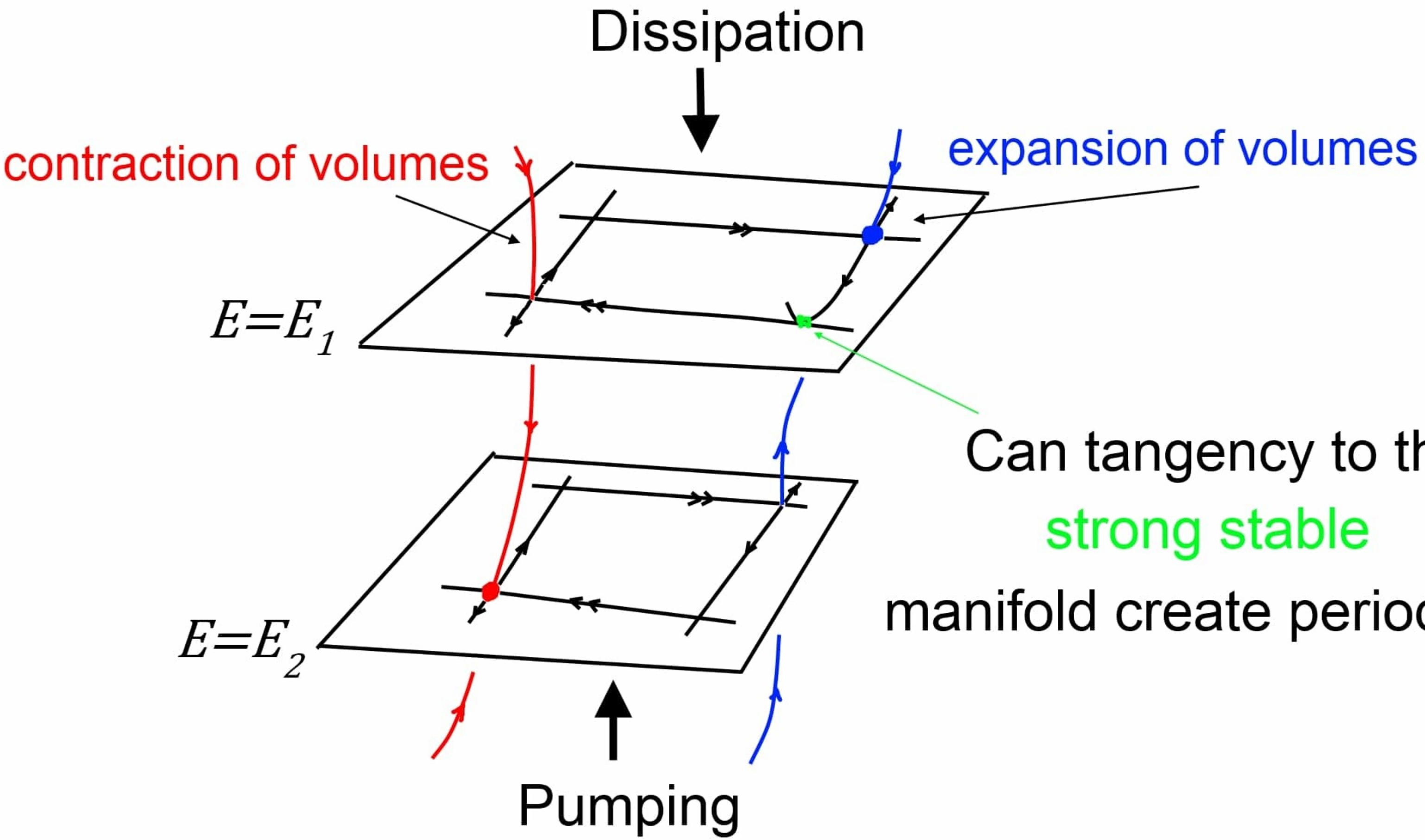
(by C^r-small perturbation, $r > 1$)

contraction of volumes



expansion of volumes

Taking dissipation into account:



Can tangency to the strong stable manifold create periodic spots?

Smooth rocks may roll on a surface in a peculiar chaotic regime (reversible core), which is an attractor and a repeller at the same time, and the complexity of fine details of this motion may (possibly) exceed everything