Examples of incompressible flows and model equations

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\[ u_t + u \nabla u + \nabla p - \nu \Delta u = 0 \]
\[ \text{div } u = 0 \]

Energy identity
\[ \int_{\Omega} \left( \frac{1}{2} |u(x,t)|^2 + \int_0^t \int_{\Omega} \nu |\nabla u(x,t')|^2 \, dt' \right) \, dx = \int_{\Omega} \frac{1}{2} |u_0(x)|^2 \, dx \]

+ localized versions

boundary cond.:
\[ u \big|_{\partial \Omega} = 0 \quad (\nu > 0) \]
\[ u \cdot n \big|_{\partial \Omega} = 0 \quad (\nu = 0) \]
Also have good linear estimates for
\[ u_t + \nabla p - \nu \Delta u = \text{div} f = \text{div} (-u \otimes u) \]

\[ n = 2 \]
Energy + linear estimates are sufficient for regularity,
( for \( \Omega \neq 0 \) this is, in fact, a “critical case”)

\[ n \geq 3 \]
Energy + linear estimates are not sufficient

all we know at present for general solutions

Could it be in this case that

what is not forbidden is allowed?

by our current knowledge
An analogy

\[ \dot{x} = \int D H(x), \quad x = (x_1, \ldots, x_{2n}) \]

Considered quantities: \( H = f_0 \cdot f_1 \cdot \ldots \cdot f_m \)

\[ f_j(x(0)) = c_j \]

\[ f_j(x(t)) = c_j \quad \forall t \quad (\text{i.e. } f_j(x(t)) \neq c_j \text{ is forbidden}) \]

a form of "ergodic hypothesis":

generically, \((x)\) is all we can say about \(x(t)\),

practically, everything else is "allowed" (in the right interpretation)
Possible reasons for an Erg. Hyp. failure:

- a missed conserved quantity \( f_{m+1}(x(t)) = f_m(t(01)) \)
  (famous example: Kovaleuskaya's spinning top)
- KAM

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Analogies in the regularity problem
- a new estimate
- more subtle reasons for regularity

Remark:
- on an abstract level
- regularity \( \Rightarrow \) estimates
- but estimates implied by regularity can be "implied"
New estimates can appear in special situations.

Examples:

- Anti-symmetric solutions: \( r \cdot u(\theta) \)
  - No swirl: \( \omega(\theta) \) \( (\text{away from 2D}) \)

- Leray's self-similar solutions \( \text{(Ne\u0103as, \u00e7i\u0163i\u0103c\u0103c, S.)} \)

\[
    u(x, t) = \frac{1}{1 - \frac{x}{T-t}} \ U \left( \frac{x}{1 - \frac{x}{T-t}} \right) : \quad \frac{1}{2} |U|^2 + x \cdot \bar{U} + P
\]

\( \uparrow \text{pressure} \)
Examples of (b)

- A class of equations (x) \( u_t + \frac{b(u_t, Du)}{b_{\text{linear, with}}} \Delta u = 0 \), \( u_i = (u_1, \ldots, u_m) \)

\[ u_t + u D u + \frac{D}{2} |u|^2 + u \nabla u \nabla u - \Delta u = 0 \]

Geodesics in Diff(N) ("pattern matching equation")

Radial solutions: \( u(x) = -x \cdot u(r, t), \quad r = 1 \times 1 \)

Steady states (by ODEs)

regularity (localized data) \( n \leq 4 \)

blow-up \( n \geq 5 \)

\( l_{21} \sim |x|^{-\frac{2}{3}} \) \( (n = 3) \)
\( \sim |x|^{-1} \log |x| \) \( (n = 4) \)
\( \sim |x|^{-1} \) \( (n \geq 5) \)

Regularity is not easily explained in terms of some quantity with a "good estimate."
- Dyadic models
(Katz-Pavlovic, Barbatis et al, Tao, Chekridou, ...)

\[ \alpha = \frac{2}{3} \sim 3 \text{dim NSE} \]

\[ \dot{X}_n = -\nu X_n + \nu^{n-1} X_{n-1}^2 - \nu^{n} X_n X_{n+1} \]

"regularity" \sim fast decay of \( X_n \) as \( n \to \infty \)

Energy \sim \sum X_n^2, energy est. similar to NSE

blow-up from smooth data only in dimensions \( n \geq 5 \)

\[ n \leq 4 \quad \rightarrow \quad |X_n| \rightarrow |X_{n+1}| \rightarrow |X_{n+2}| \rightarrow \text{energy flux} \]

\[ \text{energy flux} \]

faster-than-expected flux to higher \( n \)
enhances viscosity

\( \rightarrow \) regularity for \( n = 3 \)

similarities with previous example
Examples of situations with no "hidden estimates" (in some sense)

2d Euler in \( \Omega = B_1 = \{ x \in \mathbb{R}^2, |x| < 1 \} \)

\( u = (u_1, u_2), \quad u \cdot n = 0 \) at \( \partial \Omega \), \( \omega = \text{curl} \ u = u_{21} - u_{12} \) \( \text{div} \ u = 0 \)

\[ \omega_t + u \nabla \omega = 0 \]

\[ (*) \quad \| \omega(t) \|_{L^\infty} = \| \omega(0) \|_{L^\infty}, \quad \| \nabla u \|_{L^\infty} \lesssim \| \omega \|_{L^\infty} \]

\[ \Rightarrow \quad \| u(t) - u(\tau, t) \|_{L^\infty} \leq c \| x - \gamma \| \log \frac{1}{|x - \gamma|} \]

\( x = u(1, t) \)

\( x(t) - \gamma(t) \geq e^{-ce^ct} \)

No particle collisions \( \rightarrow \) regularity
for smooth \( \omega(0) \) \( \| D\omega \|_{L^\infty} \leq c \), similar for higher norms
\[
\text{Remark: what one really needs is } \int_0^t \| \omega(s) \|_{L^\infty} ds < +\infty \quad \text{(in 3d this is the Beale-Kato-Majda regularity crit.)}
\]
Gives 2d regularity; "critical" argument, essentially "no room to spare"
- does not survive modification of the map \( \omega \to \omega^\varepsilon \) to order \(-1+\varepsilon\)
- change from \( \omega_0 \in L^\infty \) to \( \omega_0 \in L^p \) (large \( p \))

Does not really use much the Euler equation (beyond the \( L^\infty \)-est. for \( \omega \))

Can one have a "deeper" estimate?
(analogy of a new conservation law)

Theorem (Kiselev, S.) The double exponential estimate is optimal
(when \( \omega_0 \neq 0 \)).
The flow which "saturates" the estimate (motivated by numerical investigations of Luo-Hou)

"near collision" of points

model of what happens at \( \mathbb{R} \):

\[
\omega(x) \rightarrow u(x)
\]

\( u_x = H \omega \) (Hilbert tr.)

\( \omega_t + u \omega_x = 0 \)

"1d model of 2d Euler", in some sense

behavior can be captured by ODE calculations
The full Luo-Hou calculation

\[
\begin{align*}
\omega_t + u \nabla \omega &= \Theta_x, \\
\Theta_t + u \nabla \Theta &= 0
\end{align*}
\]

3d axi-symmetric Euler

\[
\eta = \frac{\omega^{(\Theta)}}{r}, \quad f = ru^{(\Theta)}_1, \quad u = (u^{(r)}, u^{(z)})
\]

\[
\begin{align*}
\eta_t + u \nabla \eta &= \left( \frac{f^2}{r^2} \right)_z, \\
f_t + u \nabla f &= 0
\end{align*}
\]
These flows are quite close to 2d (and only “a little” super-critical).

General point: for difficult super-critical equations it is useful to study regimes “close to critical” (or transition to super-critical) at first.

“Deep super-critical” regimes are hard to control.

Rigorous results for 1d models of the 2d Boussinesq

\[ u(x, t) = (u(x, t), v(x, t)) \]

\[ u(x, 0) \rightarrow u(x) \]

\[ \omega(x, 0) \rightarrow \omega(x) \quad \text{Biot - Savart model} \]

\[ \Theta(x, 0) \rightarrow \Theta(x) \]

\[ u_x = H \omega \quad \text{Hilbert transf.} \]
$1d$ model: \((x \in \mathbb{R})\)

\[
\begin{align*}
\omega_t + u \omega_x &= \Theta_x \\
\Theta_t + u \Theta_x &= 0 \\
\omega_x &= H\omega
\end{align*}
\]

\((*)\)

"Boussinesq"

Remark: incompressible flow in $\Omega$ can produce compressible flow at $\partial \Omega$.

\textbf{Theorem} \quad (Choi, Hou, Kiselev, Luo, Yao, S.)

\((*)\) exhibits finite-time blow-up from smooth, compactly supported initial data.
Related 1d models

\[ \omega_t = u_x \omega \quad \text{(Constantin-Lax-Majda, 1980s)} \]
\[ u_x = H \omega \quad \text{finite time blow-up possible} \]

\[ \omega_t + u \omega_x = u_x \omega \quad \text{(De Gregorio, 1990s) regularity open} \]
\[ u_x = H \omega \]

\[ \omega_t + u \omega_x + 2u_x \omega = 0 \]
\[ u_x = H \omega \]
	ext{geodesics in Diff}(S^1)
with $H^{1/2}$ metric

\[ \omega_t + u \omega_x = 0 \]
\[ u = H \omega \]
(no derivative of $u$)

\text{finite-time blow-up} \quad \text{(Preston et al. 2015)}

\text{related to SQG} \quad \text{finite-time blow-up} \quad \text{(Cordoba et al., 2000s)}
Initial data

\[ \omega \]

Blow-up

\[ \theta \]

\[ Y = \int_{0}^{\infty} \log x \Theta'(x) \, dx \]

\[ \frac{d^2 Y}{dt^2} \geq c \left( 1 + Y^2 \right) \]

infinitive compression on particle collision

numerics: 1d model solutions seem to capture the 2d equations well

uses some relatively hidden monotonicity properties

Does not give blow-up rates
Another (probable) example of “no hidden estimates” (Jia, 5.)

3d NSE Cauchy problem in \( \mathbb{R}^3 \)

\[
\begin{align*}
    u_t + u \nabla u + \nabla p - \Delta u &= 0, \quad \mathbb{R}^3 \times (0, T) \\
    \text{div} \ u &= 0 \\
    u(x, 0) &= u_0(x)
\end{align*}
\]

Scaling symmetry:

\[
\begin{align*}
    u(x, t) &\rightarrow u \ u(ux, u^2 t) & u &\rightarrow u^2 \\
    u_0(x) &\rightarrow u \ u_0(ux) & u_0 &\rightarrow u^{1 \alpha} \\
    p(x, t) &\rightarrow u^2 \ p(ux, u^2 t) & p &\rightarrow p^\alpha
\end{align*}
\]
Perturbation theory approach to the Cauchy problem

\[ u(t) = e^{t\Delta} u_0 + \text{correction} \]

small for small time

locally in time, viscosity dominates

Borderline ("critical") spaces \( X \) for the argument

\[ \| u_0 \|_X = \| u_0 \|_X \] (scale-invariant norm)

\( X = H^{1/2}, L^3, B^{-1 + \frac{2}{p}}, L^p \) \( p, q \) ---

\( \tilde{X} = L^3_{\text{weak}}, BMO^{-1} \) ---

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Kato (1980s)

Cannone

Gallagher

Koch-Tataru

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short time or small norm

large \( \| u_0 \|_X \)

does not work

O.K.
Difference between "type X" and "type X":

\[ \frac{1}{|x|} \notin X \quad \text{and} \quad \frac{1}{|x|} \in \widetilde{X} \]

\( u_0 \in X \Rightarrow \text{for "type } \widetilde{X} \text{", no "hidden smallness" condition} \)

\[ u_0 \upharpoonright_{B_R} \xrightarrow{u \to 0} 0 \]

For \( u_0 \in X \) and short times \( \cdots \) sub-critical regime

For \( u_0 \in \widetilde{X} \) \( \cdots \) super-critical regime from \( t=0 \)
Heuristics: \[ u_0 \sim \frac{a}{|x|}, \quad \Delta u_0 \sim \frac{|a|}{|x|^3}, \quad u_0 \Delta u_0 \sim \frac{a^2}{|x|^3} \]

| \Delta u_0 | >> | u_0 \Delta u_0 | \quad \text{for small } a \quad \text{(perturbative regime)}

| \Delta u_0 | << | u_0 \Delta u_0 | \quad \text{for large } a \quad \text{(non-perturbative regime)}

\[ u_0 = a \cdot (-1) \quad \text{(-1-homogeneous field)} \]

Transition from perturbative to non-perturbative occurs at any time interval (no matter how short)
Analogy with the standard steady-state problem

\[-\Delta u + u \nabla u + \nabla \rho = 0\]

\[\text{div } u = 0\]

\[u \bigg|_{\partial \Omega} = kg\]

\[L_{21} u = -\Delta u + u \nabla u + u \nabla u + \nabla q\]
Scale-invariant solutions

\[ u(x,t) = \frac{1}{\sqrt{t}} \ U\left( \frac{x}{\sqrt{t}} \right) \]

\[ - \Delta \bar{U} + \frac{1}{2} \ x \cdot \nabla \bar{U} + \frac{1}{2} \ \bar{U} + \bar{U} \cdot \nabla \bar{U} + \nabla P = 0 \quad \text{in } \mathbb{R}^3 \]

\[ \text{div} \ \bar{U} = 0 \]

\[ \bar{U}(x) = k \ u_0(x) + O\left( \frac{1}{|x|} \right) \quad x \to \infty \]

\[ L_k \ \text{-- linearization at } \bar{U} \]

Theorem (Jia, S.)

The problem always has a solution (for any \( k \in \mathbb{R} \)).
Spectrum of $L_k$ (in suitable spaces)

Spectral condition (*):

The spectrum crosses for sufficiently large $k$ (and non-degeneracy).

$\overline{u}_0$ = truncation of a $(-1)$-homog. field $k u_0$ to compact support.

$\overline{u}_0$ is at the border of the perturbative regime.
Theorem (cha - 5.)

Assume (*) is satisfied (for $u_0$). Then there are smooth fields $a^{(n)}$, $b^{(n)}$ supported in $B_R$ for a fixed $R > 0$, $a^{(n)} \to u_0$, $b^{(n)} \to u_0$ in $L^{3-\varepsilon}$ for each $\varepsilon > 0$, such that the limits of the solutions $u^{(n)}(t)$ and $v^{(n)}(t)$ (with $u^{(n)}(0) = a_n$, $v^{(n)}(0) = b^{(n)}$) are different.

Corollary: (*) $\Rightarrow$ existing local well-posedness results proved by perturbation theory are essentially optimal.