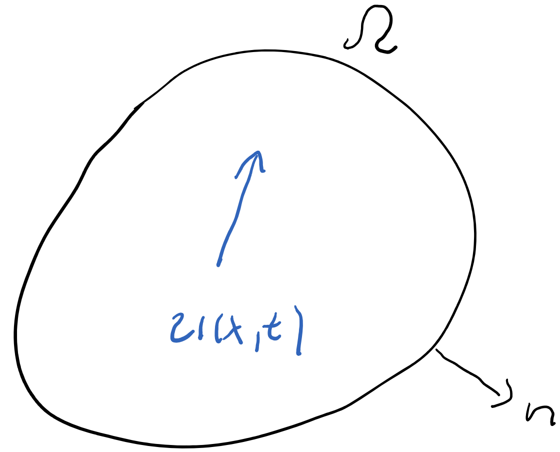
Examples of incompressible flows and model equations

Vladimir Sverak

University of Minnesota

$$u_t + u\nabla u + \nabla p - \nu \Delta u = 0$$

$$\operatorname{div} u = 0$$



Energy identity

$$\int_\Omega \frac{1}{2}|u(x,t)|^2 + \int_0^t \int_\Omega \nu |\nabla u(x,t')|^2 dt' = \int_\Omega \frac{1}{2}|u_0(x)|^2 dx$$

+ localized versions

boundary cond. :

$$u\big|_{\partial\Omega} = 0 \quad (\nu > 0)$$

$$u\cdot n\big|_{\partial\Omega} = 0 \quad (\nu = 0)$$

Also have good linear estimates for

$$u_t + \nabla p - \nu \Delta u = \operatorname{div} f = \operatorname{div}(-u \otimes u)$$

$n = 2$ Energy + linear estimates are sufficient for regularity; (for $\partial\Omega \neq 0$ this is, in fact, a "critical case")

$n \geq 3$ $\underbrace{\text{Energy + linear estimates}}_{\text{all we know at present for general solutions}}$ are not sufficient

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Could it be in this case that

what is not forbidden is allowed?

by our current knowledge

An analogy

$$\dot{x} = J \nabla H(x) \quad , \quad x = (x_1, \dots, x_{2n})$$

Consered quantities: $H = f_0, f_1, \dots, f_m$

$$f_j(x(0)) = c_j$$

(*) $f_j(x(t)) = c_j \quad \forall t$ (i.e. $f_j(x(t)) \neq c_j$ is forbidden)

a form of "ergodic hypothesis":

generically, (*) is all we can say about $x(t)$, (practically) everything else is "allowed" (in the right interpretation)

Possible reasons for an Erg. Hyp. failure:

- a missed conserved quantity $\left(f_{m+1}(x(t)) = f_{m+1}(x(0)) \right)$
  (famous example: Kovalevskaya's spinning top)

- KAM

- ....

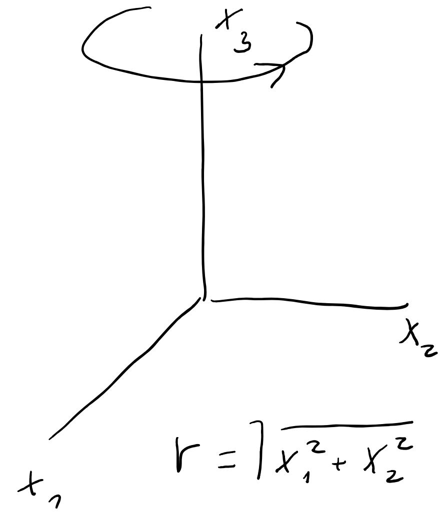
Analogies in the regularity problem

- a new estimate

- more subtle reasons for regularity

Remark:
on an abstact level
regularity $\Leftrightarrow$ estimates
but estimates implied
by regularity can be
"implicit"

New estimates can appear in special situations



$r = \sqrt{x_1^2 + x_2^2}$

Examples:

- axi-symetric solutions: $\quad r \cdot u^{(\theta)}$

(Yudovich, Ladyzhenskaya, ...) no swirl: $\quad \dfrac{\omega^{(\theta)}}{r}$ (away from $\partial\Omega$)

- Leray's self-similar solutions (Nečas, Růžička, S.)

$$u(x,t) = \frac{1}{\sqrt{T-t}}\, \bar{U}\left(\frac{x}{\sqrt{T-t}}\right) \quad : \quad \frac{1}{2}|U|^2 + x \cdot \bar{U} + P$$

pressure

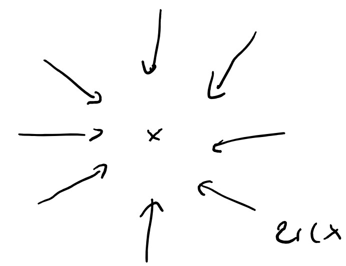
Examples of (b)

- a class of equations (*) $u_t + \underbrace{b(u, \nabla u)}_{\text{bilinear, with}} - \Delta u = 0$, $u = (u_1, \dots, u_m)$

$\int b(\varphi, \nabla\varphi)\cdot\varphi = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$

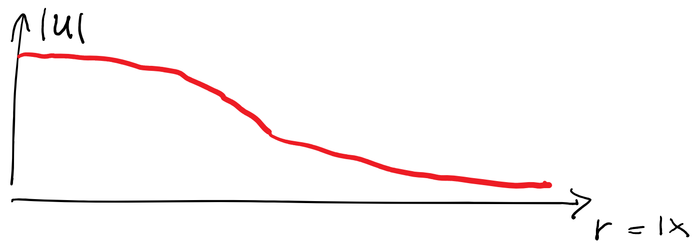
$$\underbrace{u_t + u\nabla u + \nabla \frac{|u|^2}{2} + u \operatorname{div} u}_{\text{geodesics in } \mathrm{Diff}(\Omega)} - \Delta u = 0$$

("pattern matching equation")



Radial solutions: $u(x) = -x\, v(r,t)$, $r = |x|$

Steady states
(by ODEs)



$|u| \sim |x|^{-\frac{2}{3}}$ $(n=3)$

$\sim |x|^{-1} \log |x|$ $(n=4)$

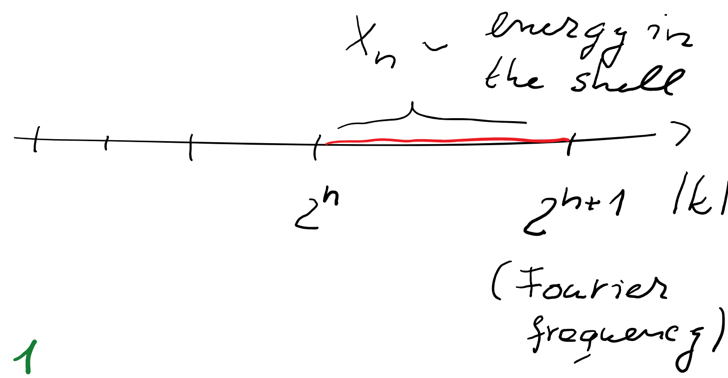
$\sim |x|^{-1}$ $(n \geq 5)$

regularity (localized data) $n \leq 4$

blow-up $n \geq 5$

regularity is not easily explained in terms of some quantity with a "good estimate"

- Dyadic models
(Katz-Pavlovic, Barbato et al, Tao, Cheskidov, ...)



$\alpha = \frac{2}{5}$ ~ 3dim NSE

$$\dot{X}_n = -\lambda^{2n\alpha} X_n + \lambda^{n-1} X_{n-1}^2 - \lambda^n X_n X_{n+1}$$

"regularity" ~ fast decay of X_n as $n \to \infty$

Energy ~ $\sum' X_n^2$, energy est. similar to NSE

blow-up from smooth data only in dimensions $n \geq 5$

$n \leq 4$ $\longrightarrow \boxed{X_n} \longrightarrow \boxed{X_{n+1}} \longrightarrow \boxed{X_{n+2}} \longrightarrow$

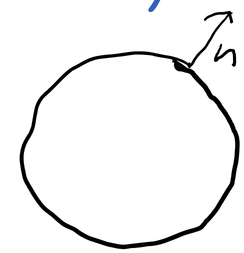
$\longrightarrow$ energy flux

faster-than-expected flux to higher n enhances viscosity

$\Rightarrow$ regularity for $n = 3$

similarities with previous example

Examples of situations with no "hidden estimates" (in some sense)



2d Euler in $\Omega = B_1 = \{x \in \mathbb{R}^2, \ |x| < 1\}$

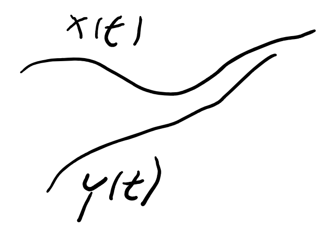
$u = (u_1, u_2)$, $u \cdot n = 0$ at $\partial\Omega$, $\omega = \operatorname{curl} u = u_{2,1} - u_{1,2}$, $\operatorname{div} u = 0$

$$\omega_t + u \nabla \omega = 0$$

$$(*) \quad \|\omega(t)\|_{L^\infty} = \|\omega(0)\|_{L^\infty}, \qquad \|\nabla u\|_{\cancel{L^\infty}\, BMO} \le c \|\omega\|_{L^\infty}$$

$$\Rightarrow \quad |u(x,t) - u(y,t)| \le c\,|x-y| \log \frac{1}{|x-y|}$$

$$\dot{x} = u(x,t) \qquad |x(t) - y(t)| \ge e^{-ce^{ct}}$$



No particle collisions $\longrightarrow$ regularity for smooth $\omega(0)$

$\|\nabla\omega\|_{L^\infty} \leq c\, e^{ce^{ct}}$, similar for higher norms

Remark: what one really needs is $\int_0^t \|\omega(s)\|_{L^\infty} ds < +\infty$ (in 3d this is the Beal-Kato-Majda regularity crit.)

Gives 2d regularity; "critical" argument, essentially "no room to spare" to order $-1+\varepsilon$

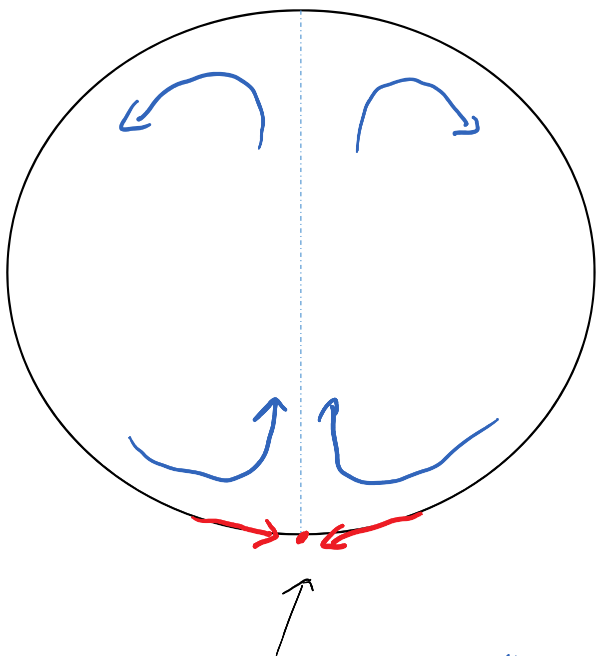
- does not survive modification of the map $\omega \longrightarrow u$
- — " — change from $\omega_0 \in L^\infty$ to $\omega_0 \in L^p$ (large p)

Does not really use much the Euler equation (beyond the L^∞-est. for ω)

Can one have a "deeper" estimate?
(analogy of a new conservation law)

Theorem (Kiselev, S.) The double exponential estimate is optimal (when $\partial\Omega \neq \emptyset$).

The flow which "saturates" the estimate (motivated by numerical investigations of Luo-Hou)



"near collision" of points

model of what happens at $\partial\Omega$:

$x \in \mathbb{R}$

$\omega(x) \longrightarrow u(x)$

$u_x = H\omega$ (Hilbert tr.)

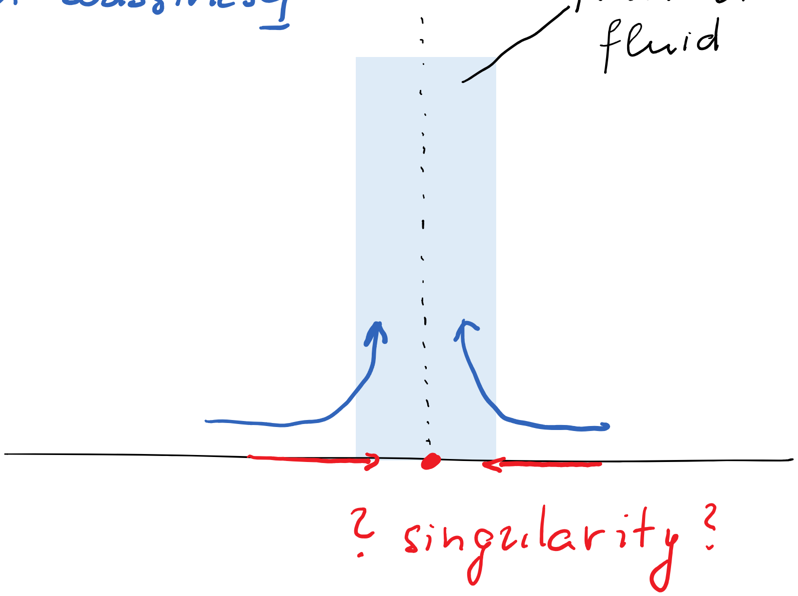
$\omega_t + u\omega_x = 0$

"1d model of 2d Euler", in some sense

behavior can be captured by ODE calculations

The full Luo - Hou calculation

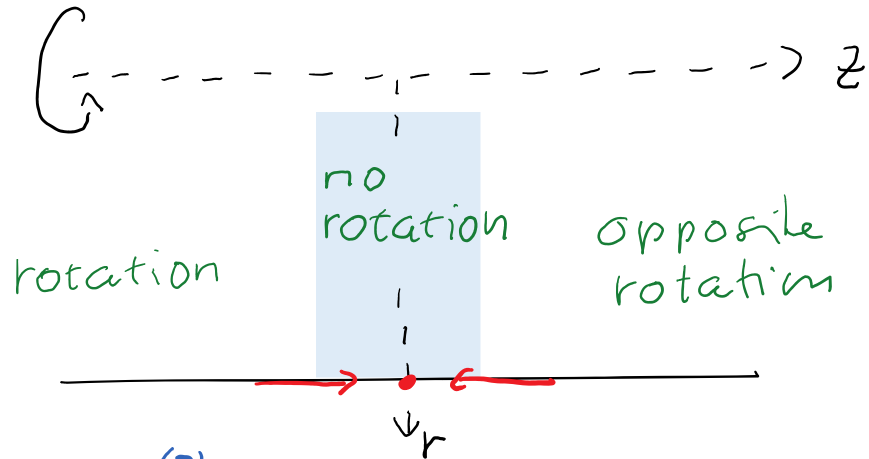
2d Boussinesq



$$\omega_t + u \nabla \omega = \theta_{x_1}$$

$$\theta_t + u \nabla \theta = 0$$

3d axi-symmetric Euler



$$\eta = \frac{\omega^{(\theta)}}{r}, \quad f = r u^{(\theta)}, \quad u = (u^{(r)}, u^{(z)})$$

$$\eta_t + u \nabla \eta = \left(\frac{f^2}{r^2}\right)_{,z}$$

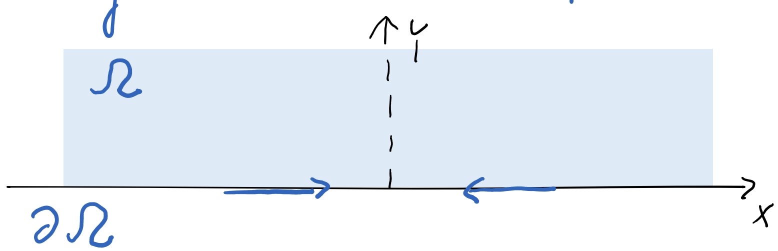
$$f_t + u \nabla f = 0$$

These flows are quite close to 2d (and only "a little" super-critical)

General point: for difficult super-critical equations it is useful to study regimes "close to critical" (or transition to super-critical) at first.

"Deep super-critical" regimes are hard to control.

Rigorous results for 1d models of the 2d Boussinesq



Dynamics at $\partial\Omega$?

$$\mathbf{u}(x,y) = (u(x,y), v(x,y))$$

$$u(x,0) \longrightarrow u(x)$$
$$\omega(x,0) \longrightarrow \omega(x)$$
$$\theta(x,0) \longrightarrow \theta(x)$$

Biot-Savart model

$$u_x = H\omega$$

Hilbert transf.

1d model: $(x \in \mathbb{R})$

$$\left.\begin{aligned} \omega_t + u\,\omega_x &= \theta_x \\ \theta_t + u\,\theta_x &= 0 \\ u_x &= H\omega \end{aligned}\right\} \ (*) \ \text{"Boussinesq"}$$

Compare with

$$\left.\begin{aligned} \omega_t + u\,\omega_x &= 0 \\ u_x &= H\omega \end{aligned}\right\} \ \text{"Euler"}$$

(captures well the real 2d situation)

Remark: incompressible flow in Ω can produs compressible flow at $\partial\Omega$.

Theorem (Choi, Hou, Kiselev, Luo, Yao, S.)

(*) exhibits finite-time blow-up from smooth, compactly supported initial data.

Related 1d models

$$\omega_t = u_x \omega \qquad u_x = H\omega$$

(Constantin-Lax-Majda, 1980s)

finite time blow-up possible

$$\omega_t + u\omega_x = u_x \omega \qquad u_x = H\omega$$

(De Gregorio, 1990s) regularity open

$$\omega_t + u\omega_x + 2u_x\omega = 0 \qquad u_x = H\omega$$

geodesics in $\mathrm{Diff}(S^1)$ with $\dot{H}^{1/2}$ metric

finite-time blow-up (Preston et al 2015)

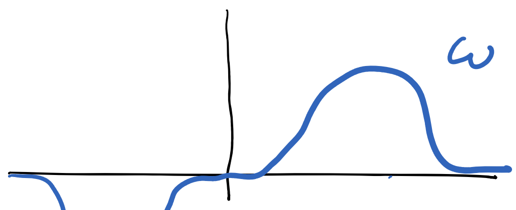
$$\omega_t + u\omega_x = 0 \qquad u = H\omega$$

(no derivative of u)

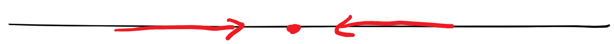
related to SQG

finite-time blow-up (Cordoba et al, 2000s)

Initial data

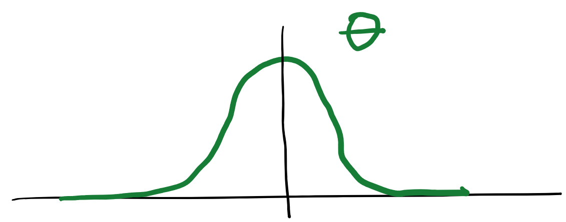


Blow-up



infinite compression
or particle collision

numerics: 1d model
solutions seem to capture
the 2d equations well



$$\Psi = \int_0^\infty \log x \, \theta'(x)\, dx$$

$$\frac{d^2 \Psi}{dt^2} \geq c\,(1 + \Psi^2)$$

uses some relatively hidden
monotonicity properties
Does not give blow-up rates

Another (probable) example of "no hidden estimates" (Jia, S.)

3d NSE Cauchy problem in $\mathbb{R}^3$

$$\left.\begin{aligned} u_t + u\nabla u + \nabla p - \Delta u &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \mathbb{R}^3 \times (0,T)$$

$$u(x,0) = u_0(x)$$

Scaling symmetry:

$$u(x,t) \longrightarrow \lambda u(\lambda x, \lambda^2 t) \qquad u \to u_\lambda$$

$$u_0(x) \longrightarrow \lambda u_0(\lambda x) \qquad u_0 \to u_{0\lambda}$$

$$p(x,t) \longrightarrow \lambda^2 p(\lambda x, \lambda^2 t) \qquad p \to p_\lambda$$

Perturbation theory approach to the Cauchy problem

$$u(t) = e^{t\Delta} u_0 + \underbrace{\text{correction}}_{\text{small for small time}}$$

locally in time, viscosity dominates

Borderline ("critical") spaces X for the argument

(Kato (1980s), Cannone, Gallagher, Koch-Tataru, ...)

$$\| u_{0\lambda} \|_X = \| u_0 \|_X \qquad (\text{scale-inv. norm})$$

$$X = \dot{H}^{1/2},\ L^3,\ B^{-1+\frac{2}{p}}_{p,q},\ \dots$$

$$\tilde{X} = L^3_{weak},\ BMO^{-1},\ \dots$$

short time (large $\|u_0\|_X$) → does not work

or

small norm (global solutions) → O.K.

Difference between "type X" and "type $\tilde{X}$":

$$\frac{1}{|x|} \notin X \quad , \quad \frac{1}{|x|} \in \tilde{X}$$

$u_0 \in X \Rightarrow$ $\qquad$ for "type $\tilde{X}$", no "hidden smallness" condition

$$u_{0,\lambda}\Big|_{B_R} \xrightarrow{\lambda \to 0} 0$$

For $u_0 \in X$ and short times ---- sub-critical regime

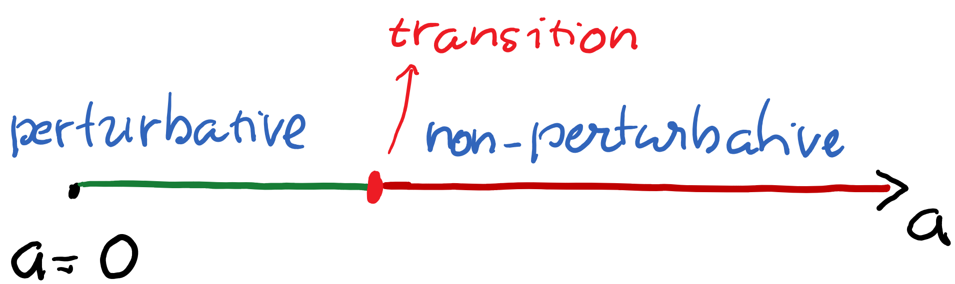
For $u_0 \in \tilde{X}$ ... super-critical regime from $t=0$

Heuristics: $\quad u_0 \sim \frac{a}{|x|}, \quad \Delta u_0 \sim \frac{a}{|x|^3}, \quad u_0 \nabla u_0 \sim \frac{a^2}{|x|^3}$

$$|\Delta u_0| \gg |u_0 \nabla u_0| \quad \text{for small } a \quad \text{(perturbation regime)}$$

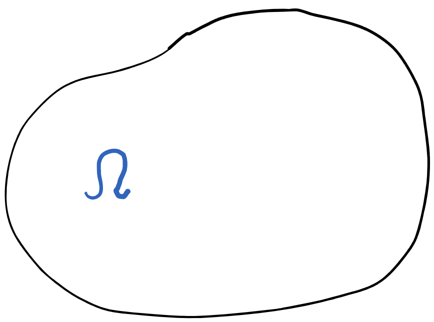
$$|\Delta u_0| \ll |u_0 \nabla u_0| \quad \text{for large } a \quad \text{(non-perturbative regime)}$$

$$u_0 = a \cdot (-1\text{-homogeneous field})$$



any time interval
(no matter how short)

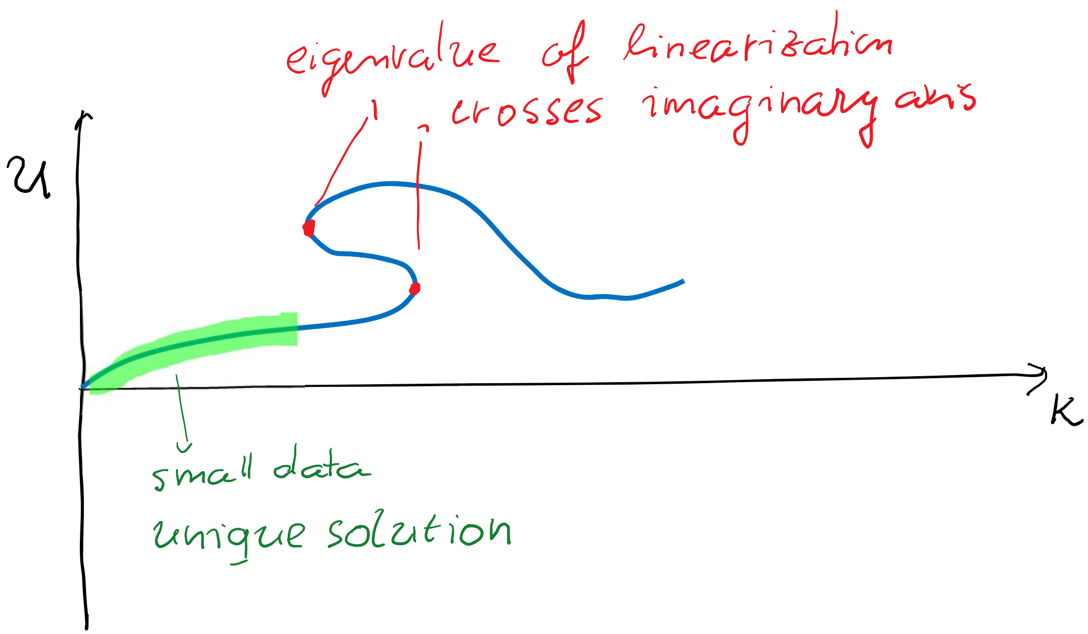
Analogy with the standard steady-state problem



$$-\Delta u + u\nabla u + \nabla p = 0$$

$$\operatorname{div} u = 0$$

$$u\big|_{\partial\Omega} = \kappa g$$



$$L_u v = -\Delta v + u\nabla v + v\nabla u + \nabla \underline{q}$$

Scale-invariant solutions

$$u(x,t) = \frac{1}{\sqrt{t}} \bar{U}\left(\frac{x}{\sqrt{t}}\right);$$

$$-\Delta \bar{U} + \frac{1}{2} x \cdot \nabla \bar{U} + \frac{1}{2} \bar{U} + \bar{U} \cdot \nabla \bar{U} + \nabla P = 0 \quad \text{in } \mathbb{R}^3$$

$$\operatorname{div} \bar{U} = 0$$

$$\bar{U}(x) = k\, u_0(x) + O\left(\frac{1}{|x|}\right)$$

$$x \longrightarrow \infty$$

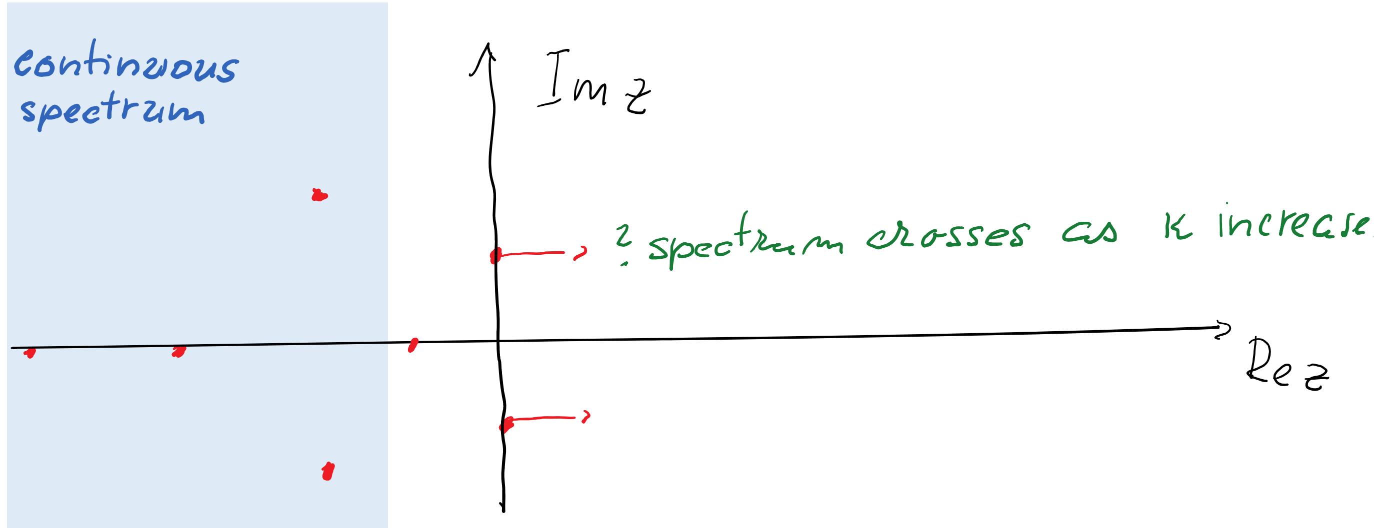
L_k .... linearization at $\bar{U}$

Theorem (Jia, S.)

The problem always has a solution (for any $k u_0(x)$).

Spectrum of L_k

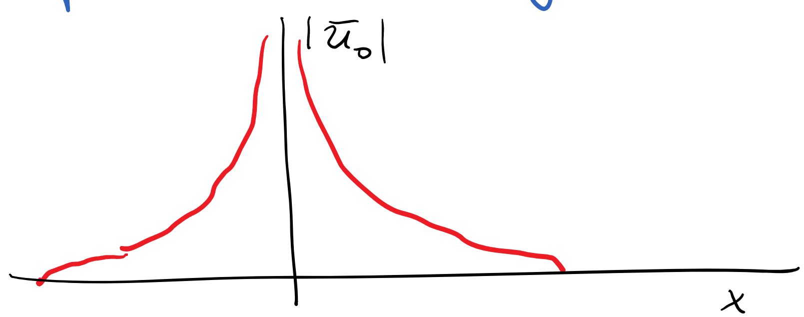
(in suitable spaces)



Spectral condition (*):

The spectrum crosses for sufficiently arge k (+ non-degeneracy).

$\bar{u}_0$ = truncation of a (-1)-homog. field κu_0 to compact support.



$\bar{u}_0$ is at the border of the perturbative regime

Theorem (Jia - S.)

Assume (*) is satisfied (for $\bar{u}_0$). Then there are smooth fields $a^{(n)}, b^{(n)}$ supported in B_R for a fixed $R>0$, $a^{(n)} \longrightarrow \bar{u}_0$, $b^{(n)} \longrightarrow \bar{u}_0$ in $L^{3-\varepsilon}$ for each $\varepsilon>0$, such the the limits of the solutions $u^{(n)}(t)$ and $v^{(n)}(t)$ (with $u^{(n)}(0)=a_n$, $v^{(n)}(0)=b^{(n)}$) are different.

Corollary: (*) $\Rightarrow$ existing local well-posedness results proved by perturbation theory are essentially optimal.