# Energy concentration and type II blow up

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## Wave propagation

- Various physical contexts: nonlinear optics, plasma physics, fluid mechanics, ferromagnetism, astrophysics...
- Competition between two phenomenons:
  – dispersion or dissipation: the wave tends to spread/dissipate during propagation.
  – concentration: nonlinear interaction with the medium (focusing laser beams, gravitational force, ...)
- One canonical model: Nonlinear Schrödinger equation,

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u_{|t=0}(x) = u_0(x) \ \text{ smooth} \end{cases} \qquad x \in \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}.$$

## Qualitative description

- Local existence well understood (1980's)
- Existence of special solutions (stationary, periodic, travelling waves).
  – ODE's, calculus of variations.
  – Special nonlinear waves: solitary waves.
- Long time asymptotics behavior of solutions:
  – asympotic generic behavior: scattering, soliton resolution problem.
  – interaction.
  – blow up and concentration of energy.

## Conservation laws and structure

$$i\partial_t u + \Delta u + u|u|^{p-1} = 0, \quad x \in \mathbb{R}^d.$$

- Conservation laws:

$$\left\{ \begin{array}{l} \text{Energy}: \quad E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} = E(u_0) \\ \text{Mass}: \quad \int |u|^2 = \int |u_0|^2 \end{array} \right.$$

- Scaling symmetry:

$$\left| \begin{array}{l} u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0, \\ \|\nabla^{s_c} u_\lambda(t,\cdot)\|_{L^2} = \|\nabla^{s_c} u(\lambda^2 t,\cdot)\|_{L^2} \;\; \text{for} \;\; s_c = \frac{d}{2} - \frac{2}{p-1}. \end{array} \right.$$

- Critical space is $\dot{H}^{s_c}$.

# Critical and super critical problems

- $s_c = 0$: mass critical case.
  – smallest nonlinearity for which blow up is possible.
  – critical space is $L^2$: blow up happens by concentration of the mass.

- $s_c = 1$: energy critical case.
  – borderline case
  – relevant for some geometrical models (wave maps, Schrödinger maps, ...)

- $s_c > 1$: energy super critical case.
  – little known
  – conservation laws control weak norms.

## The blow up problem

Problem: describe mechanisms of energy concentration/singularity formation.

- Heat equation: [Giga, Kohn 1985], [Herrero,Velasquez, 92], [Matano, Merle 04], [Mizoguchi 06], maximum principle for the scalar problem.

- Dispersive equations:
  – Semilinear wave/(NLS) equations: [John 1975], [Alinhac 90'], [Martel, Merle 2000], [Perelman 00], [Merle, R. 01], [Krieger, Schlag, Tataru 07], [Merle, Zaag 08-12], [Merle, R., Rodnianski 14],
  – General relativity and compressible fluids [Christodoulou 10].

## The energy super critical NLS problem

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u_{|t=0}(x) = u_0(x) \ \ \text{smooth} \end{cases} \qquad x \in \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}.$$

We consider the energy super critical range

$$s_c = \frac{d}{2} - \frac{2}{p-1} > 1.$$

Problem Description of blow up bubbles:

- Smooth well localized initial data, robust construction
- Genericity/stability of the blow up bubble

## Self similar profile

Look for solutions of the form

$$u(t,x)=\frac{1}{\lambda(t)^{\frac{2}{p-1}}}\Phi\left(\frac{x}{\lambda(t)}\right),\quad \lambda(t)=\sqrt{T-t}$$

then

$$(*)\quad \Delta\Phi+i\Lambda\Phi+\Phi|\Phi|^{p-1}=0,\quad \Lambda\Phi=\frac{2}{p-1}\Phi+y\cdot\nabla\Phi.$$

- Singular homogeneous solution: $\Phi^*=\frac{c_\infty}{r^{\frac{2}{p-1}}}$.
- Regular solution: for the heat, requires $p<p_{JL}$.

From now on,

$$p>p_{JL}=1+\frac{4}{d-4-2\sqrt{d-1}}\quad \text{(implies } d\geq 11\text{)}.$$

$\Longrightarrow$ Expectation: no self similar blow up.

## Solitary wave profile

- Solitary wave: $u(t,x) = Q(x)$,

$$\begin{cases} Q'' + (d-1)\frac{Q'}{r} + Q^p = 0 \\ Q(0) = 1, \quad Q'(0) = 0 \end{cases}$$

- Asymptotic behavior (ODE's):

$$Q(r) \sim \frac{c_\infty}{r^{\frac{2}{p-1}}} = \Phi^* \quad \text{as} \quad r \to +\infty.$$

- $Q \notin H^1(\mathbb{R}^d)$: very bad stationary solution.

# Type II blow up

[Merle, R., Rodnianski 14] Let $p > p_{JL}$. There exist $\mathcal{C}^\infty$ compactly supported data such that

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) e^{i\gamma(t)}$$

with

$$\lambda(t) \sim (T-t)^{\frac{k}{\alpha}}, \quad \alpha = \alpha(d,p) > 0, \quad k \in \mathbb{N}^*, \quad k > \frac{\alpha}{2}.$$

– Previous works for the heat [Herrero, Velasquez 92], [Matano. Merle 04], [Mizoguchi 06]: based on Lyapounov functionals induced by the maximum principle.
– Finite codimensional stability, [Collot 14] (wave).
– Related problems: Schrödinger/wave maps, harmonic heat flow, . . . , critical cases.

## Perturbation of the solitary wave

- The solitary wave is stable by really small perturbations:

$$u_0 = Q + \varepsilon_0, \quad \|\varepsilon_0\|_{H^1} \ll 1 \ \text{ implies } \ T = +\infty.$$

  – [Burq, Planchon, Stalker, Tahvildar-Zadeh 04]
  – Infinite energy initial data.

- Description of the finite energy blow up bubble

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(Q+\varepsilon)\left(t, \frac{x}{\lambda(t)}\right) e^{i\gamma(t)}$$

then

$$\lim_{t\to T} \|u(t)\|_{\dot{H}^s} \begin{cases} = +\infty \text{ for } \ s > s_c, \\ < +\infty \ \text{ for } \ s < s_c \end{cases}, \qquad \text{Type II.}$$

## The flow near the solitary wave

- General problem: describe the flow near the solitary wave
- [Nakanishi, Schlag 10] for (NLS), [Martel, Merle, R.,10-13] for (gKdV) $s_c = 0$, complete description of the soliton instability:
  – blow up (stable) with a unique blow up speed
  – scattering ie linear behavior as $t \to +\infty$ (stable)
  – soliton behavior as $t \to +\infty$ (threshold).
  $\Longrightarrow$ Minimal critical elements.
- More blow up speeds [Krieger, Schlag, Tataru 07], [Perelman 12], [Martel, Merle, R. 12] : threshold dynamics.

## Heart of the analysis

step 1 Construction of an approximate solution

– Derivation of an ODE for scaling

– Quantization of blow up speeds.

step 2 Control of the infinite dimensional part

– Energy method (non radial also)

– Essential role of scaling and super critical norms.

Illustration on a slightly simpler model: the radial Stefan problem.

# Exterior Stefan problem

- Melting of an ice ball (no surface tension): the temperature $u : \Omega(t) \to \mathbb{R}$ evolves according to:

$$\begin{cases} \partial_t u - \Delta u = 0 \quad \text{in} \quad \Omega(t) \\ \partial_n u = V_{\Gamma(t)} \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega(t). \end{cases}$$

- Spherical symmetry: for $\Omega(t) = \{x \in \mathbb{R}^2;\ |x| \geq \lambda(t)\}$ and $x \in \mathbb{R}^2$:

$$\begin{cases} u_t - u_{rr} - \frac{1}{r} u_r = 0 \quad \text{in} \quad \Omega(t) \\ u_r(t, \lambda(t)) = -\dot{\lambda}(t), \quad u(t, \lambda(t)) = 0 \end{cases}$$

- free boundary problem: melting/cooling and concentration of energy on the boundary?

# Melting/cooling regimes

[Hadzic, R. 15]: There exist finite time melting regimes:

$$\lambda(t) \sim_{t\to T} \begin{cases} (T-t)^{1/2} e^{-\frac{\sqrt{2}}{2}\sqrt{|\ln(T-t)|}}, & \text{stable} \\ \frac{(T-t)^{\frac{k+1}{2}}}{|\log(T-t)|^{\frac{k+1}{2k}}}, \quad k \in \mathbb{N}^*, & \text{codimension k} \end{cases}$$

– pioneering work [Herrero, Velazquez 00]
– connection to [R., Schweyer 10] on the heat flow.

[Hadzic, R. 15]: There exist finite time cooling regimes:

$$\lambda(t) - \lambda_\infty \sim_{t\to+\infty} \frac{1}{t^{k+1}(\log t)^2}, \quad \text{codimension k}, \quad k \in \mathbb{N}$$

– duality between melting/cooling.

# Renormalization

- Renormalization

$$u(t,r) = v(s,y), \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \quad y = \frac{r}{\lambda(t)}$$

so that

$$\begin{cases} \partial_s v - \Delta v + a(s) y \partial_y v = 0, \quad y \geq 1, \quad a = -\frac{1}{\lambda}\frac{d\lambda}{ds} \\ v(s,1) = 0, \quad \partial_y v(s,1) = a \end{cases}$$

- Boundary is fixed $y = 1$

Problem: extract the dynamical system for $a$ with

$$a(s) \to 0 \quad \text{as} \quad s \to +\infty,$$ type II concentration.

# Spectral problem

$$\left\{\begin{array}{l} -\Delta + by\partial_y, \\ v(1) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} -\Delta + z\partial_z, \\ v(\sqrt{b}) = 0 \end{array}\right. .$$

$\Longrightarrow$ Thin boudary layer $z \sim \sqrt{b} \ll 1$.

- On $\mathbb{R}^2$ with radial symmetry, spectral basis

$$(-\Delta + z\partial_z)P_k = \lambda_k P_k, \quad \lambda_k = 2k, \quad P_k = \text{Laguerre polynomial.}$$

- Lyapounov Schmidt like argument (singular):

$$\left\{\begin{array}{l} (-\Delta + z\partial_z)\psi_{b,k} = \lambda_{b,k}\psi_{b,k} \\ \psi_{b,k}(\sqrt{b}) = 0 \end{array}\right. , \quad \lambda_{b,k} \sim 2k + \frac{2}{|\log b|}.$$

$\Longrightarrow$ Attention: $\psi_{b,k}(z) \sim z^k$ for $z \gg 1$!

## Approximate solution

Inject an ansatz

$$v(s,y) \sim \Sigma_{j=0}^{k} b_j(s)\psi_{b,j}(y) \text{ into } \partial_s v - \Delta v + a(s)y\partial_y v = 0,$$

then projecting onto the eigenmodes leads to the dynamical system:

$$\left| \begin{array}{l} \frac{db_j}{ds} + bb_j\left(2j + \frac{2}{|\log b|}\right) = 0, \quad \text{eigenvalue equation} \\ a = -\frac{1}{\lambda}\frac{d\lambda}{ds}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2} \quad \text{scaling law} \\ b_s + 2b(b-a) = 0, \quad \text{time dependance of the operator} \\ a = \Sigma_{j=0}^{k} b_j\left(1 + \frac{2}{|\log b|}\right), \quad \text{dynamical boundary condition} \end{array} \right.$$

$\Longrightarrow$ ODE's driving melting/concentration.

## Exact solution

Inject an ansatz

$$v(s,y) = \Sigma_{j=0}^{k} b_j(s)\psi_{b(s),j}(y) + \varepsilon(s,y).$$

- Close energy estimates on $\varepsilon$ using sharp spectral gap estimates in weighted spaces.
- Treat the time dependance of the operator.
- Need to use derivatives to close ($H^2$ theory): nonlinear algebra.

$\Longrightarrow$ Sharp use of dissipation and the parabolic structure.

## Conclusion and perspectives

- Understanding of some instability mechanisms near the solitary wave in parabolic and dispersive problems.
- Parabolic setting: refined energy method using sharp spectral gap estimates.
- Extension to the non radial case in progress.
- First steps towards: classification, more complicated problems.