Mixing and enhanced dissipation in the inviscid limit of the Navier-Stokes equations near the 2D and 3D Couette flows

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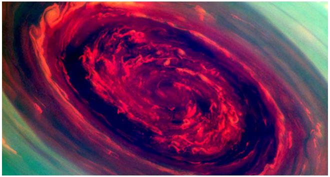

Navier-Stokes system

A *viscous* incompressible fluid is governed by the Navier-Stokes system :

$$\partial_t v + v \cdot \nabla v + \nabla p = \nu \Delta u \tag{1a}$$

$$\nabla \cdot v = 0, \tag{1b}$$

where $v \in \mathbb{R}^D$ is the velocity, p is the pressure and $\nu = \mathbf{Re}^{-1}$ denotes the inverse Reynolds number.

We will be working in 2D or 3D:

- In the 3D setting: $(x, y, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$.
- In the 2D setting: $(x, y) \in \mathbb{T} \times \mathbb{R}$.

Couette flow

- Consider a *viscous* incompressible fluid flowing with velocity $(y, 0, 0)$ (sometimes called the 'Couette flow').

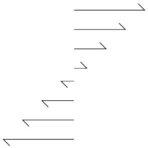

Figure : The simplest non-zero fluid motion imaginable

- We want to know what happens if we make a very small disturbance to this flow, e.g : a/ does the fluid motion settle back down,
  b/ does it oscillate periodically
  c/ or does it go completely turbulent?
- We will be studying this in the simplest 3D setting: $(x, y, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ or 2D setting $(x, y) \in \mathbb{T} \times \mathbb{R}$.

Hydrodynamic stability

Understanding the stability of laminar flows and the transition to turbulence is one of the main objectives of hydrodynamic stability theory :

- One of the first and most influential experiments in the field were those of Reynolds in 1883, which demonstrated the instability of the laminar flow in a pipe for sufficiently high Reynolds number.
- However, such instabilities appeared inconsistent with theoretical studies, which suggested spectral stability independent of Reynolds number for a variety of simple laminar flows, including variations of the Couette flow

This leads to the so-called Sommerfeld paradox or turbulence paradox:

One of the first main explanation was given by W. Orr (1866-1934) in 1907 and is based on transient growth, now usually refered to as the Orr mechanism.

There are other explanations : lift-up effect, secondary instability, *subcritical transition* or *by-pass transition*.

Transition threshold

- One way to address this paradox is to observe that while the flow is technically stable for all finite Reynolds number, the set of stable perturbations shrinks as the Reynolds number increases.
- One can try to determine how the maximal size of stable perturbations in a given norm, the "transition threshold" , will scale with respect to the viscosity :
- It will be important to note that the transition threshold *depends on the norm* and that different norms may result in different answers. One way to see this is that at high Reynolds numbers, the viscosity will not suppress the high frequencies and echoes (resonances) have more time to yield growth.

Transition threshold

- Goal: for a given norm N, find a $\gamma = \gamma(N)$ such that

$$\|u_{in}\|_N \ll \textbf{Re}^{-\gamma} \Rightarrow \text{ asymptotic stability}$$
$$\|u_{in}\|_N \gtrsim \textbf{Re}^{-\gamma} \Rightarrow \text{ possible instability.}$$

- This γ is usually called the *transition threshold*.
- A great deal of work has been devoted to estimating γ for various laminar flow configurations (Orszag et. al., Trefthen et. al., Waleffe, Chapman,Reddy et. al. and *many* others...)
- For the problem we are considering, estimates range in $1 \leq \gamma \leq 7/4$.

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- For the problem we are considering, estimates range in $1 \leq \gamma \leq 7/4$.
- We prove: (A) for N sufficiently strong (Gevrey class) that $\gamma = 1$ and (B) we identify the only possible instability for disturbances of size $\mathbf{Re}^{-1} \lesssim \|u_{in}\|_N \lesssim \mathbf{Re}^{-2/3+\delta}$.

This is a 3D effect. Indeed, for sufficiently regular perturbations, the 2D Couette flow is nonlinearly, asymptotically stable (in Gevrey spaces) uniformly at high Reynolds number $\gamma = 0$.

We study the 3D Navier-Stokes equations near the Couette flow in the idealized domain $(x, y, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$: if $u + (y, 0, 0)^T$ solves the Navier-Stokes equation, then the disturbance u solves

$$\partial_t u + y\partial_x u + u \cdot \nabla u + \nabla p = \begin{pmatrix} -u^2 \\ 0 \\ 0 \end{pmatrix} + \nu \Delta u \tag{2a}$$

$$\nabla \cdot u = 0, \tag{2b}$$

where $\nu = \mathbf{Re}^{-1}$ denotes the inverse Reynolds number,
p can be split into two parts : $p = p^{NL} + p^L$ where

$$\Delta p^{NL} = -\partial_i u^j \partial_j u^i \tag{3a}$$

$$\Delta p^L = -2\partial_x u^2 \tag{3b}$$

p^{NL} is the nonlinear contribution to the pressure due to the disturbance and p^L is the linear contribution to the pressure due to the interaction between the disturbance and the Couette flow.

The linearized problem

Let us start by the linearized problem.

At the linear level, one can uderstand the paradox by the fact that the linearization of NS around Couette flow is non-normal, which means a large transient growth before eventual decay

- The suggestion that this is the source of the observed instability goes back to Orr in 1907, even though he was thinking about a 2D non-normal effect called the *Orr mechanism*, which will not be the main cause of transient growth in 3D. Indeed, in 2D we have asymptotically stability (in a suitable sense) uniformly at high Reynolds number.
- In 3D, the main mechanism for transient kinetic energy growth is the 3D non-normal effect known as the *lift-up effect*. The work of Trefethen et. al. (TTRD93) forwarded the idea that the nonlinearity could interact poorly with the non-normal behavior by repeatedly re-exciting growing linear modes, producing a “nonlinear bootstrap” scenario.

Four or five main linear effects:

- We have two (stabilizing) linear effects: *inviscid damping* and *mixing-enhanced dissipation*, which are both a result of the fluid *mixing itself*. There is also the regular viscosity for the zero modes.
- There are two destabilizing linear effects in 3D : *lift-up effect and vorticity stretching*.
- Of course there is also the destabilizing effect of the nonlinearity.
- The main proof will be to make sure that the (stabilizing) linear effects are strong enough to overcome the destabilizing effects and the nonlinearity.

The linearized system reads :

$$\partial_t u + y\partial_x u = \begin{pmatrix} -u^2 \\ 0 \\ 0 \end{pmatrix} - \nabla p^L + \nu \Delta u \tag{4a}$$

$$\Delta p^L = -2\partial_x u^2 \tag{4b}$$

$$\nabla \cdot u = 0. \tag{4c}$$

Let us start by the 2D case and write the system in vorticity formulation $\omega = \partial_x u^2 - \partial_y u^1$:

Linearization in 2D: vorticity formulation

- ω solves:

$$\omega_t + y\partial_x\omega = \nu\Delta\omega \tag{5}$$

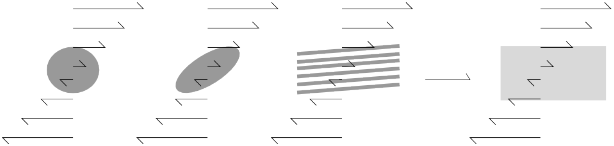
$$\Delta\psi = \omega \quad u = \nabla^{\perp}\psi. \tag{6}$$

- The linear problem was explicitly solved by Kelvin in 1887.
- The inviscid case, $\nu = 0$, was studied more carefully by Orr in 1907.
- His analysis is now known as the *Orr mechanism*.

The Orr mechanism I: mixing

- Consider the inviscid case ($\nu = 0$), where the solution is

$$\omega(t, x, y) = \omega_{in}(x - ty, y)$$



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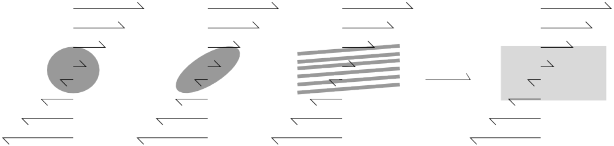

- Taking the Fourier transform:

$$\hat{\omega}(t, k, \eta) = \hat{\omega}_{in}(k, \eta + kt).$$

- Linear-in-time transfer of information to high frequencies!
- Weakly converges back to equillibrium:

$$\omega(t) \rightharpoonup < \omega_{in} >_x = \frac{1}{2\pi} \int \omega_{in}(x, y) dx.$$

- Hence *mixing* is an *infinite dimensional* phenomenon,

The Orr mechanism II: inviscid damping

- Kelvin and Orr used the change of coordinates

$$z = x - ty$$
$$f(t,z,y) = \omega(t,x,y)$$
$$\phi(t,z,y) = \psi(t,x,y),$$

- In the inviscid case $f(t,z,y) = \omega_{in}(z,y)$.
- ϕ satisfies

$$\partial_{zz}\phi + (\partial_y - t\partial_z)^2\phi = f$$

$$\hat{\phi}(k,\eta) = -\frac{\hat{f}(k,\eta)}{k^2 + |\eta - kt|^2}.$$

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- We see the fundamental *decay-by-mixing*:

$$\|\phi_{k\neq 0}(t)\|_{H^N} \lesssim \langle t\rangle^{-2}\|f\|_{H^{N+2}} = \langle t\rangle^{-2}\|\omega_{in}\|_{H^{N+2}}.$$

- *Damping costs regularity.*

The Orr mechanism II: Inviscid damping and transient linear growth

- After undoing the change of coordinates we get the *inviscid damping*:

$$\|u^1_{k\neq 0}(t)\|_2 + \langle t\rangle \|u^2(t)\|_2 \lesssim \langle t\rangle^{-1}.$$

- The velocity converges *strongly* back to a shear flow!

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- Orr understood where the regularity loss comes from: consider a pure plane wave with $\eta \gg k$:

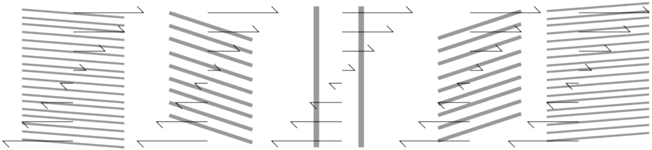

Figure : The center image occurs at the *critical time* $t = \eta/k$ (based on a picture of Boyd). Information can just as easily *unmix* as it can mix!

Viscous dynamics: enhanced dissipation

- In the viscous case f satisfies

$$\partial_t \hat{f}(t,k,\eta) = -\nu \left(|k|^2 + |\eta - kt|^2 \right) \hat{f}(t,k,\eta)$$

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- We get Kelvin's solution:

$$\hat{f}(t,k,\eta) = \hat{\omega}_{in}(k,\eta) \exp\left[-\nu \int_0^t |k|^2 + |\eta - k\tau|^2 \, d\tau \right].$$

- If $k \neq 0$, then $\int_0^t |\eta - k\tau|^2 \, d\tau \gtrsim \min(\eta^2 t, k^2 t^3)$
- By the time $\nu t^3 \gtrsim 1$, almost all of the $k \neq 0$ modes are wiped out:

$$\|\omega_{k\neq 0}(t)\|_2 \lesssim e^{-c\nu t^3}.$$

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- *Enhanced dissipation* since it occurs on $O(\nu^{-1/3})$ time scales rather than the heat equation $O(\nu^{-1})$ time scale.
- Summary: the shear flow is transferring vorticity to high frequencies at a linear rate in time, greatly enhancing the viscous term.

Gevrey classes. Maurice Gevrey (1884-1957)

In 1918, Maurice Gevrey defined the following:

Definition: Let $m \geq 1$. $G^m(\mathbb{T})$ (*Gevrey space of class m*) is the set of $f = f(x)$ s.t.

$$\exists C, \tau > 0, \quad |f^{(k)}(x)| \leq C\tau^{-k}(k!)^m, \quad C, \tau > 0, \quad \forall k, x.$$

Remark:

- $m = 1$: analytic functions.
- $m > 1$: $G^m(\mathbb{T})$ contains compactly supported functions.

Proposition: $f \in G^m(\mathbb{T})$ iff

$$\exists C, \sigma > 0, \quad |\hat{f}(k)| \leq Ce^{-\sigma k^{1/m}}$$

Gevrey norms

- In physical space

$$\|f\|^2_{G_\sigma^{\tau;\frac{1}{m}}} := \sum_{j\in N} \left(\tau^j (j!)^{-m} j^\sigma\right)^2 \|\partial^j f\|^2_{L^2} \tag{7}$$

- In Fourier space

$$\|f\|^2_{\mathcal{G}_\sigma^{\tau;\frac{1}{m}}} := \||\xi|^\sigma e^{\tau|\xi|^{1/m}} \hat{f}(\xi)\|^2_{L^2} \tag{8}$$

m is the Gevrey class
σ is a Sobolev correction
τ is the *radius* of analyticity when $m = 1$.

Gevrey norms (mostly for analytic regularity) are used in many PDE problems :

- Temam-Foias
- Bardos-BenAchour
- Ferrari-Titi
- Levermore-Oliver-Titi
- Sammartino and Caflisch
- Kukavica-Temam-Vicol-Ziane
- Rauch

The inviscid limit near 2D Couette flow

Theorem (Bedrossian,Masmoudi,Vicol '14)

For $s \in (1/2, 1)$, $\lambda > \lambda' > 0$, $\delta > 0$ and all integers $\alpha \geq 1$ there exists ϵ_0, K_0 (independent of ν) such that if $\omega_{in}^\nu = \omega_S^\nu + \omega_R^\nu$ is mean-zero and

$$\int |y\omega_{in}^\nu(x,y)|\, dxdy + \|\omega_S^\nu\|_{\mathcal{G}^{\lambda;s}} + e^{K_0\nu^{-\frac{(3+\delta)s}{2(1-s)}}}\|\omega_R^\nu\|_2 = \epsilon \leq \epsilon_0, \tag{9}$$

then for all ν sufficiently small (independent of ϵ) the solution $\omega^\nu(t)$ satisfies the following with all constants independent of ν and t:

$$\|\omega_{k\neq 0}^\nu(t, x + ty + \Phi(t,y), y)\|_{\mathcal{G}^{\lambda';s}} \lesssim \frac{\epsilon}{\langle \nu t^3\rangle^\alpha} \tag{10}$$

$$\|\omega_0^\nu(t,y)\|_{\mathcal{G}^{\lambda';s}} \lesssim \frac{\epsilon}{\langle \nu t\rangle^{1/4}} \tag{11}$$

$$\|u_{k\neq 0}^1(t)\|_2 + \langle t\rangle \|u^2(t)\|_2 \lesssim \frac{\epsilon}{\langle t\rangle\langle \nu t^3\rangle^\alpha}, \tag{12}$$

where

$$\Phi(t,y) = \int_0^t e^{\nu(t-\tau)\partial_{yy}} < u^1 > (\tau, y) d\tau; \tag{13}$$

The linearized inviscid equation in 3D reads :

$$\partial_t u + y\partial_x u = \begin{pmatrix} -u^2 \\ 0 \\ 0 \end{pmatrix} - \nabla p^L \tag{14a}$$

$$\Delta p^L = -2\partial_x u^2 \tag{14b}$$

$$\nabla \cdot u = 0. \tag{14c}$$

it has long been known that the quantity

$$q^2 = \Delta u^2,$$

plays in dimension 3 a similar role to that played by the vorticity in dimension 2. This unknown dates back at least to Lord Kelvin. It solves :

$$\partial_t q^2 + y\partial_x q^2 = 0. \tag{15}$$

If we rewind by the action of the Couette flow and define $X = x - ty$, write $U^i(t, X, y, z) = u^i(t, x, y, z)$, and $Q^2(t, X, y, z) = q^2(t, x, y, z)$ and $P^L(t, X, y, z) = p^L(t, x, y, z)$, then we derive

$$\partial_t U = \begin{pmatrix} -U^2 \\ 0 \\ 0 \end{pmatrix} - \nabla^L P^L \tag{16a}$$

$$\partial_t Q^2 = 0 \tag{16b}$$

$$\Delta_L U^2 = Q^2 \tag{16c}$$

$$\Delta_L P^L = -2\partial_X U^2 \tag{16d}$$

$$\nabla^L \cdot U = 0, \tag{16e}$$

where we are using

$$\nabla^L = (\partial_X, \partial_y - t\partial_X, \partial_z) \tag{17a}$$

$$\Delta_L = \partial_{XX} + (\partial_y - t\partial_X)^2 + \partial_{zz}. \tag{17b}$$

Here 'L' stands for 'linear'.

We have from the elementary inequality $\frac{1}{k^2+(\eta-tk)^2} \lesssim \frac{\langle\eta\rangle^2}{\langle kt\rangle^2}$ for any non-zero integer k, the following fundamental *inviscid damping* estimate for any $\sigma \in [0,\infty)$ and $\beta \in [0,2]$,

$$\|\Delta_L^{-1} f_{\neq}\|_{H^\sigma} = \left(\sum_{l,k\neq 0} \int \frac{\langle k,\eta,l\rangle^{2\sigma} \left|\hat{f}(k,\eta,l)\right|^2}{\left(k^2+l^2+|\eta-kt|^2\right)^2} d\eta \right)^{1/2} \lesssim \frac{1}{\langle t\rangle^\beta} \|f_{\neq}\|_{H^{\sigma+\beta}}, \quad (18)$$

where we use the notation :

$$f_0(y,z) = \frac{1}{2\pi} \int f(x,y,z)dx, \quad (19a)$$

$$f_{\neq} = f - f_0, \quad (19b)$$

where then '$\neq$' refers to the projection to non-zero Fourier frequencies in x. In particular this yields the decay of U^2:

Linearized dynamics: inviscid damping

$$\|U^2_{\neq}\|_{H^N} \lesssim \langle t\rangle^{-2}\|(Q^2_{in})_{\neq}\|_{H^{N+2}}.$$

This shows that the background shear flow suppresses x variations in u^2 even at infinite Reynolds number. In turn, this implies the inviscid damping of the linear pressure P^L:

$$\|P_L\|_{H^N} \lesssim \langle t\rangle^{-2}\|U^2_{\neq}\|_{H^{N+3}} \lesssim \langle t\rangle^{-4}\|(Q^2_{in})_{\neq}\|_{H^{N+5}}.$$

Hence, we see that $U^1_{\neq}$ and $U^3_{\neq}$ actually converge strongly as $t \to \infty$. We can therefore infer that in general, there is no inviscid damping on $u^1_{\neq}$ and $u^3_{\neq}$.

Linearized dynamics: Lift-up effect

Next, we observe that the only contribution on the RHS of the linearized equation which is *not* integrable in time is the X average of U^2. Indeed, upon taking X averages in X, we derive the degenerate Jordan block-type system

$$\partial_t U_0^1 = -U_0^2 \tag{20a}$$

$$\partial_t U_0^2 = \partial_t U_0^3 = 0. \tag{20b}$$

By (20), U_0^1 grows linearly in time, and therefore the 3D Couette flow is linearly (algebraically) unstable in the 3D Euler equations (although classically known to be *spectrally stable* in the sense that there are no unstable eigenvalues). Hence, we see that the instability is "non-modal".

Linearized dynamics: vortex stretching

- Given that U^2 will rapidly decay independent of Reynolds number, we can integrate the momentum equations (in the new coordinates) for U^1, U^3 to deduce (I am putting the effect of viscosity back)

$$\|U^1_{\neq}\|_{H^\sigma} \lesssim \|U\|_{H^{\sigma+5}} e^{-c\nu t^3}$$
$$\|U^3_{\neq}\|_{H^\sigma} \lesssim \|U\|_{H^{\sigma+5}} e^{-c\nu t^3}.$$

- This is sharp: there is no inviscid damping on U^1 or U^3 – in fact for times $1 \ll t \ll \nu^{-1/3}$, $U^{1,3}$ are almost constant in time.
- There is a direct cascade of *kinetic energy* as u^1 and u^3 are sent to smaller and smaller scales.

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- There is a direct cascade of *kinetic energy* as u^1 and u^3 are sent to smaller and smaller scales.
- Due to vortex stretching – hence this is a 3D effect.
- In 2D, $U^1_{\neq}$ also experiences inviscid damping and the linear solution returns to equillibrium uniformly in Reynolds number

Linearized dynamics: vortex stretching

If we take $q = \Delta u$, we can also see the effect of the linear vortex stretching :

$$\begin{cases} \partial_t q^1 + y\partial_x q^1 + 2\partial_{xy} u^1 = -q^2 + 2\partial_{xx} u^2 + \nu\Delta q^1 \\ \partial_t q^2 + y\partial_x q^2 = \nu\Delta q^2 \\ \partial_t q^3 + y\partial_x q^3 + 2\partial_{xy} u^3 = 2\partial_{zx} u^2 + \nu\Delta q^3. \end{cases}$$

In the new coordinate system : $X = x - ty$, $Y = y$ and $Z = z$:

$$\begin{cases} Q^1_t = -Q^2 - 2\partial^L_{XY} U^1 + 2\partial_{XX} U^2 + \nu\Delta_L Q^1 \\ Q^2_t = \nu\tilde{\Delta}_L Q^2 \\ Q^3_t = -2\partial^L_{XY} U^3 + 2\partial^L_{XZ} U^2 + \nu\Delta_L Q^3. \end{cases} \tag{21}$$

$$\nabla^L = (\partial_X, \partial_Y - t\partial_X, \partial_Z) \tag{22}$$

$$\Delta_L = \partial_{XX} + (\partial_Y - t\partial_X)^2 + \partial_{ZZ}. \tag{23}$$

Proposition (Linearized Euler)

Let u_{in} be a divergence free vector field with $u_{in} \in H^7$. Then the solution $u(t)$ to the linearized Euler equations (14) with initial data u_{in} satisfies the following for some final state $u_\infty = (u^1_\infty, 0, u^3_\infty)$:

$$\|u^2_{\neq}(t)\|_2 + \|u^2_{\neq}(t, x+ty, y, z)\|_{H^3} \lesssim \langle t\rangle^{-2}\|u^2_{in}\|_{H^7} \tag{24a}$$

$$\|u^1_{\neq}(t, x+ty, y, z) - u^1_\infty(x, y, z)\|_{H^1} \lesssim \langle t\rangle^{-1}\|u_{in}\|_{H^7} \tag{24b}$$

$$\|u^3_{\neq}(t, x+ty, y, z) - u^3_\infty(x, y, z)\|_{H^1} \lesssim \langle t\rangle^{-3}\|u_{in}\|_{H^7}, \tag{24c}$$

and the formulas

$$u^1_0(t, y, z) = u^1_{in\ 0}(y, z) - tu^2_{in\ 0}(y, z) \tag{25a}$$

$$u^2_0(t, y, z) = u^2_{in\ 0}(y, z) \tag{25b}$$

$$u^3_0(t, y, z) = u^3_{in\ 0}(y, z). \tag{25c}$$

Unlike the 2D case, viscosity is really necessary to stabilize the growth coming from the lift-up effect.

When accounting for finite Reynolds number we are now considering the linearized Navier-Stokes equations

$$\partial_t u + y\partial_x u = \begin{pmatrix} -u^2 \\ 0 \\ 0 \end{pmatrix} - \nabla p^L + \nu \Delta u \tag{26a}$$

$$\Delta p^L = -2\partial_x u^2 \tag{26b}$$

$$\nabla \cdot u = 0. \tag{26c}$$

From (26), we can derive the mixing enhanced dissipation, as observed by Lord Kelvin in 1887: As the Couette flow mixes information to small scales, the viscous dissipation has an increasing effect on the solution.

To understand the origins of this effect, consider the evolution of $q^2 = \Delta u^2$:

$$\partial_t q^2 + y\partial_x q^2 = \nu \Delta q^2,$$

which, after re-writing in the variables (X, y, z) with $X = x - ty$ and $Q^2(t, X, y, z) = q^2(t, X + ty, y, z)$, becomes

$$\partial_t Q^2 = -\nu \Delta_L Q^2$$
$$\partial_t \widehat{Q}^2(k, \eta, l) = -\nu(k^2 + (\eta - kt)^2 + l^2)\widehat{Q}^2(k, \eta, l),$$

which integrates to

$$\widehat{Q}^2(t, k, \eta, l) = \exp\left[-\nu \int_0^t (k^2 + (\eta - k\tau)^2 + l^2)\, d\tau\right] \widehat{Q^2_{in}}(k, \eta, l).$$

The elementary inequality $\int_0^t (k^2 + (\eta - k\tau)^2 + l^2)\, d\tau \gtrsim t^3$ for k a non-zero integer gives a decay $\sim e^{-c\nu t^3}$ for some $c > 0$ for all modes which depend on X.

Proposition (Linearized Navier-Stokes)

Let u_{in} be a divergence free vector field with $u_{in} \in H^7$. The solution to the linearized Navier-Stokes equations $u(t)$ with initial data u_{in} satisfies the following for some $c \in (0, 1/3)$

$$\|u^2_{\neq}(t)\|_2 + \|u^2_{\neq}(t, x + ty, y, z)\|_{H^3} \lesssim \langle t\rangle^{-2} e^{-c\nu t^3} \|u^2_{in}\|_{H^7} \tag{27a}$$

$$\|u^1_{\neq}(t, x + ty, y, z)\|_{H^1} \lesssim e^{-c\nu t^3} \|u_{in}\|_{H^7} \tag{27b}$$

$$\|u^3_{\neq}(t, x + ty, y, z)\|_{H^1} \lesssim e^{-c\nu t^3} \|u_{in}\|_{H^7}, \tag{27c}$$

and the formulas

$$u^1_0(t, y, z) = e^{\nu t\Delta} \left(u^1_{in\,0} - t u^2_{in\,0}\right) \tag{28a}$$

$$u^2_0(t, y, z) = e^{\nu t\Delta} u^2_{in\,0} \tag{28b}$$

$$u^3_0(t, y, z) = e^{\nu t\Delta} u^3_{in\,0}. \tag{28c}$$

there are two important time-scales: the mixing dissipation time scale $O(\nu^{-1/3})$ and the slow dissipation time scale $O(\nu^{-1})$. After $O(\nu^{-1/3})$, the x dependence of the solution has essentially been completely damped, and the evolution is dominated by the simpler (linearized) streak evolution (28).

Particular solutions of NSE: Streaks

The streaks are particular solutions of our original system which do not depend on x; one can verify that in this case, our origianl system (2) reduces to :

- $(u^2(t, y, z), u^3(t, y, z))$ solves the 2D Navier-Stokes system,
- $u^1(t, y, z)$ solves a forced, linear advection-diffusion equation.

Solutions of this general type are sometimes called "2.5 dimensional" and we will refer to this particular family as *streaks*. That is:

Proposition (Streak solutions)

Let $\nu \in [0,\infty)$, $u_{in} \in H^{5/2+}$ be divergence free and independent of x, that is, $u_{in}(x,y,z) = u_{in}(y,z)$, and denote by $u(t)$ the corresponding unique strong solution to (2) with initial data u_{in}. Then $u(t)$ is global in time and for all $T > 0$, $u(t) \in L^\infty((0,T); H^{5/2+}(\mathbb{R}^3))$. Moreover, the pair $(u^2(t), u^3(t))$ solves the 2D Navier-Stokes/Euler equations on $(y,z) \in \mathbb{R} \times \mathbb{T}$:

$$\partial_t u^i + (u^2, u^3) \cdot \nabla u^i = -\partial_i p + \nu \Delta u^i \tag{29a}$$

$$\partial_y u^2 + \partial_z u^3 = 0, \tag{29b}$$

and u^1 solves the (linear) forced advection-diffusion equation

$$\partial_t u^1 + (u^2, u^3) \cdot \nabla u^1 = -u^2 + \nu \Delta u^1. \tag{30}$$

Theorem

(Bedrossian-Germain-Masmoudi 2015) For all $s \in (1/2, 1)$, all $\lambda_0 > \lambda' > 0$, all integers $\alpha \geq 10$, all $\delta_1 > 0$, and all $\nu \in (0, 1]$, there exists constants $c_{00} = c_{00}(s, \lambda_0, \lambda', \alpha, \delta_1)$ and $K_0 = K_0(s, \lambda_0, \lambda')$ (both independent of ν), such that for all $c_0 \leq c_{00}$ and $\epsilon < c_0 \nu$, if $u_{in} \in L^2$ is a divergence-free vector field that can be written $u_{in} = u_S + u_R$ with $\|u_S\|_{\mathcal{G}^{\lambda;s}} + e^{K_0 \nu^{-\frac{3s}{2(1-s)}}} \|u_R\|_{H^3} < \epsilon$, then the unique, classical solution $u(t)$ to (2) with initial data u_{in} is global in time and the following estimates hold with all implicit constants independent of ν, ϵ, t and c_0:

(i) *transient growth of the streak: if $t < \frac{1}{\nu}$,*

$$\|u_0^1(t) - \left(e^{\nu t \Delta}\left(u_{in\ 0}^1 - t u_{in\ 0}^2\right)\right)\|_{\mathcal{G}^{\lambda';s}} \lesssim c_0^2 \tag{31a}$$

$$\|u_0^2(t) - e^{\nu t \Delta} u_{in\ 0}^2\|_{\mathcal{G}^{\lambda';s}} + \|u_0^3(t) - e^{\nu t \Delta} u_{in\ 0}^3\|_{\mathcal{G}^{\lambda';s}} \lesssim c_0 \epsilon \tag{31b}$$

Moreover, we have

(ii) uniform bounds and decay of the background streak

$$\|u_0^1(t)\|_{\mathcal{G}^{\lambda',s}} \lesssim \min\left(\epsilon\langle t\rangle, c_0\right) \tag{32a}$$

$$\|u_0^2(t)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon}{\langle \nu t\rangle^{\alpha}} \tag{32b}$$

$$\|u_0^3(t)\|_{\mathcal{G}^{\lambda',s}} \lesssim \epsilon \tag{32c}$$

$$\|u_0^1(t)\|_4 \lesssim \frac{c_0}{\langle \nu t\rangle^{1/4}} \tag{32d}$$

$$\|u_0^3(t)\|_4 \lesssim \frac{\epsilon}{\langle \nu t\rangle^{1/4}}; \tag{32e}$$

(iii) the rapid convergence to a streak

$$\|u_{\neq}^1(t, x+ty+t\psi(t,y,z), y, z)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon\langle t\rangle^{\delta_1}}{\langle \nu t^3\rangle^{\alpha}} \tag{33a}$$

$$\|u_{\neq}^2(t, x+ty+t\psi(t,y,z), y, z)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon}{\langle t\rangle^{2-\delta_1}\langle \nu t^3\rangle^{\alpha}}, \tag{33b}$$

$$\|u_{\neq}^3(t, x+ty+t\psi(t,y,z), y, z)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon}{\langle \nu t^3\rangle^{\alpha}}. \tag{33c}$$

Here $\psi(t,y,z)$ is an $O(c_0)$ correction to the mixing which depends on the disturbance and satisfies

$$\|\psi(t) - u_0^1(t)\|_{\mathcal{G}^{\lambda',s}} \lesssim \epsilon\langle t\rangle^{-1}. \tag{34}$$

Outline of the proof: New dependent variables

Take $q^i = \Delta u^i$ for $i = 1, 2, 3$. A computation shows that (q^i) solves

$$\begin{cases} \partial_t q^1 + y\partial_x q^1 + u \cdot \nabla q^1 + 2\partial_{xy} u^1 = -q^2 + 2\partial_{xx} u^2 - q^j \partial_j u^1 + \partial_x \left(\partial_i u^j \partial_j u^i\right) - 2\partial_i u^j \partial_{ij} u^1 + \nu \Delta q^1 \\ \partial_t q^2 + y\partial_x q^2 + u \cdot \nabla q^2 = -q^j \partial_j u^2 + \partial_y \left(\partial_i u^j \partial_j u^i\right) - 2\partial_i u^j \partial_{ij} u^2 + \nu \Delta q^2 \\ \partial_t q^3 + y\partial_x q^3 + u \cdot \nabla q^3 + 2\partial_{xy} u^3 = 2\partial_{zx} u^2 - q^j \partial_j u^3 + \partial_z \left(\partial_i u^j \partial_j u^i\right) - 2\partial_i u^j \partial_{ij} u^3 + \nu \Delta q^3. \end{cases}$$

Outline of the proof: New independent variables

We start with the ansatz

$$\begin{cases} X = x - ty - t\psi(t,y,z) \\ Y = y + \psi(t,y,z) \\ Z = z, \end{cases}$$

(this is motivated by a further requirement that Δ^{-1} have good properties in the new coordinates.) Consider the simple convection diffusion equation on a passive scalar $f(t,x,y,z)$

$$\partial_t f + y\partial_x f + u \cdot \nabla f = \nu \Delta f.$$

Denoting $F(t,X,Y,Z) = f(t,x,y,z)$ and $U(t,X,Y,Z) = u(t,x,y,z)$, and Δ_t and ∇^t for the expressions for Δ and ∇ in the new coordinates, this becomes

$$\partial_t F + \begin{pmatrix} u^1 - t(1+\partial_y\psi)u^2 - t\partial_z\psi u^3 - \frac{d}{dt}(t\psi) + \nu t\Delta\psi \\ (1+\partial_y\psi)u^2 + \partial_z\psi u^3 + \partial_t\psi - \nu\Delta\psi \\ u^3 \end{pmatrix} \cdot \nabla_{X,Y,Z} F = \nu\tilde{\Delta}_t F, \tag{35}$$

where $\tilde{\Delta}_t$ is a variant of Δ_t.

Eliminating u_0^1 leads to the equation

$$u_0^1 - t(1+\partial_y\psi)u_0^2 - t\partial_z\psi u_0^3 - \frac{d}{dt}(t\psi) + \nu t\Delta\psi = 0.$$

We now recast this equation on ψ in terms of $C(t,Y,Z) = \psi(t,y,z)$ and an auxiliary unknown $g = \frac{1}{t}(U_0^1 - C)$ (this roughly measures the time-oscillations of C). A variety of cancellations which take advantage of the precise structures give

$$\begin{cases} \partial_t C + \tilde{U}_0 \cdot \nabla_{Y,Z} C = g - U_0^2 + \nu\tilde{\Delta}_t C, \\ \partial_t g + \tilde{U}_0 \cdot \nabla_{Y,Z} g = -\frac{2}{t} g - \frac{1}{t}\left(U_{\neq}\cdot\nabla^t U_{\neq}^1\right)_0 + \nu\tilde{\Delta}_t g, \end{cases} \tag{36}$$

where $\tilde{U} = \begin{pmatrix} U_{\neq}^1 - t(1+\psi_y)U_{\neq}^2 - t\psi_z U_{\neq}^3 \\ (1+\psi_y)U_{\neq}^2 + \psi_z U_{\neq}^3 + g \\ U^3 \end{pmatrix}.$

We also derive in the new coordinates ($Q(t, X, Y, Z) = q(t, x, y, z)$).

$$\begin{cases} Q^1_t + \tilde{U} \cdot \nabla_{X,Y,Z} Q^1 = -Q^2 - 2\partial^t_{XY} U^1 + 2\partial_{XX} U^2 + \nu \tilde{\Delta}_t Q^1 + NL_1 \\ Q^2_t + \tilde{U} \cdot \nabla_{X,Y,Z} Q^2 = \nu \tilde{\Delta}_t Q^2 + NL_2 \\ Q^3_t + \tilde{U} \cdot \nabla_{X,Y,Z} Q^3 = -2\partial^t_{XY} U^3 + 2\partial^t_{XZ} U^2 + \nu \tilde{\Delta}_t Q^3 + NL_3, \end{cases} \tag{37}$$

We will perform most of our estimates on this system coupled with the system on the coordinates: C, g

Nonlinear terms:

We introduce the followig splitting of the linear and nonlinear terms :

$$\tilde{U} \cdot \nabla Q^\alpha = \text{“transport nonlinearity”} \qquad \mathcal{T} \qquad (38a)$$

$$-Q^j \partial_j^t U^\alpha - 2\partial_i^t U^j \partial_{ij}^t U^\alpha = \text{“nonlinear stretching”} \qquad NLS \qquad (38b)$$

$$\partial_\alpha^t(\partial_i^t U^j \partial_j^t U^i) = \text{“nonlinear pressure”} \qquad NLP \qquad (38c)$$

$$-2\partial_{XY}^t U^\alpha = \text{“linear stretching”} \qquad LS \qquad (38d)$$

$$2\partial_{X\alpha}^t U^2 = \text{“linear pressure”} \qquad LP \qquad (38e)$$

The pressure terms are named due to the fact that they arise originally from p^{NL} (in the nonlinear case) and p^L (in the linear case) in (2). The stretching terms originally arose from $\Delta(u \cdot \nabla u^\alpha)$ (in the nonlinear case) and $\Delta(y\partial_x u^\alpha)$ (in the linear case).

From the linear analysis, we expect that:

- During the early times, $t \lesssim \nu^{-1/3}$, the solution has fully 3D nonlinear effects until the enhanced dissipation eventually dominates.
- During the middle times, $\nu^{-1/3} \lesssim t \lesssim \nu^{-1}$, the solution is mostly in x-independent modes and is slowly growing via the lift-up effect.
- By the time $t \gtrsim \nu^{-1}$ the solution has, in general, become extremely large relative to ν but it is also very close to a globally regular x-independent streak and eventually returns to Couette.

The goal is to prove that for the middle and later times, the solution retains this special structure.

There are several nonlinear mechanisms which have the potential to cause instability and many have been proposed as important in the applied mathematics and physics literature for understanding transition, see e.g. Craik 1971,TTRD 1993, Reddy-Schmid 1998, Schmid-Henningson 2001 We are particularly worried about so called "bootstrap" mechanisms (see Trefethen 2005, Waleffe 1995): nonlinear interactions that repeatedly excite growing linear modes.

We classify the main effects by the x frequency of the interacting functions:

- **(2.5NS)** $(0 \cdot 0 \to 0)$ For *2.5D Navier-Stokes*, this corresponds to self-interactions of the streak.
- **(SI)** $(0 \cdot \neq \to \neq)$ For *secondary instability*, this effect is the transfer of momentum from the large u_0^1 mode to other non-zero modes .
- **(3DE)** $(\neq \cdot \neq \to \neq)$ For *three dimensional echoes*, these effects are 3D variants of the 2D hydrodynamic echo phenomenon, namely nonlinear interactions of x-dependent modes forcing unmixing modes (see Morrison 1998, Vanneste 2002,BM13). For $t \gtrsim \nu^{-1/3}$, this effect should be wiped out by the enhanced dissipation.
- **(F)** $(\neq \cdot \neq \to 0)$ For *nonlinear forcing*, this is the effect of the forcing from x-dependent modes back into x-independent modes. Similar to **(3DE)**, this effect is over-powered by the enhanced dissipation after $t \gtrsim \nu^{-1/3}$.

Choice of the norms

The choice of the norms is extremely delicate and amounts to describing precisely the possible distribution of information in Fourier space for Q and C. The highest norms are derived from a toy model. Each Q^i is measured with a slightly different norm, of the form $\|A^i(t,\nabla)Q^i(t)\|_2$ where $A^i(t,\nabla)$ are special Fourier multipliers. Let us just describe the norm used to measure Q^3,

$$A^3_k(t,\eta,\ell) = e^{\lambda(t)|k,\eta,l|^s}\langle k,\eta,l\rangle^\sigma \frac{e^{\mu|\eta|^{1/2}}}{w(t,\eta)w_L(t,k,\eta,l)}\left(\mathbf{1}_{k\neq 0}\min\left(1,\frac{\langle \eta,l\rangle^2}{t^2}\right)+\mathbf{1}_{k=0}\right).$$

We now comment on the different components:

- $e^{\lambda(t)|k,\eta,\ell|^s}$ corresponds to a Gevrey-$\frac{1}{s}$ norm, with decreasing radius,
- $\langle k,\eta,\sigma\rangle^\sigma$ gives a Sobolev correction (mainly for technical convenience).
- The factor w comes from a toy model that estimates the "worst-possible" growth of high frequencies due to weakly nonlinear effects. Roughly speaking, it is taken to satisfy the following for $|k|^2 \lesssim |\eta|$ (hence $\sqrt{|\eta|} \lesssim t \lesssim |\eta|$),

$$\frac{\partial_t w(t,\eta)}{w(t,\eta)} \sim \frac{1}{1+\left|t-\frac{\eta}{k}\right|}, \qquad \text{when } \left|t-\tfrac{\eta}{k}\right| \lesssim \tfrac{\eta}{k^2} \quad \text{and} \quad w(1,\eta)=1.$$

- The last factor corresponds to a growth occurring for times large compared to the frequency due to the linear vortex stretching. That Q^1 and Q^3 ultimately grow at least quadratically is evident on the linear level (ignoring viscosity).

Dynamics near the subcritical transition

Theorem (Above threshold dynamics)

(Bedrossian-Germain-Masmoudi 2015) For all $s \in (1/2, 1)$*, all integers* $\alpha \geq 10$*, all* $\delta \gg \delta_1 > 0$ *sufficiently small, and all* $\nu \in (0, 1]$*, there exists a norm* $N = N(s, \nu)$ *and a constant* $c_{00} = c_{00}(s, \alpha, \delta, \delta_1)$ *(independent of* ν*), such that for all* $c_0 \leq c_{00}$ *and* $\epsilon < \nu^{2/3+\delta}$*, if* $u_{in} \in L^2$ *is divergence-free and* $\|u_{in}\|_N < \epsilon$*, then* $u(t)$ *exists at least until* $t = c_0\epsilon^{-1}$ *and following holds with all constants independent of* ν*,* ϵ*,* t *and* c_0 *(for* $t \geq 1$*):*

$$\|u_0^1(t) - e^{\nu t\Delta}\left(u_{in\ 0}^1 - tu_{in\ 0}^2\right)\|_{\mathcal{G}^{\lambda',s}} \lesssim c_0^2 \tag{39a}$$

$$\|u_0^{2,3}(t)\|_{\mathcal{G}^{\lambda',s}} + \langle t\rangle^{-1}\|u_0^1(t)\|_{\mathcal{G}^{\lambda',s}} \lesssim \epsilon \tag{39b}$$

$$\|u_{\neq}^{1,3}(t, x + ty + t\Phi(t, y, z), y, z)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon\langle t\rangle^{\delta_1}}{\langle \nu t^3\rangle^{\alpha}} \tag{39c}$$

$$\|u_{\neq}^{2}(t, x + ty + t\Phi(t, y, z), y, z)\|_{\mathcal{G}^{\lambda',s}} \lesssim \frac{\epsilon}{\langle t\rangle\langle \nu t^3\rangle^{\alpha}}, \tag{39d}$$

where $\Phi(t, y, z)$ *is an* $O(c_0)$ *correction to the mixing.*

Discussion of result

- This shows that that the secondary instability of a streak is the only possible instability (*for sufficiently smooth data*).
- The 2/3 threshold is predicted by weakly nonlinear analysis. A toy model predicts that the inviscid nonlinear time-scale is $\tau_{NL} \gtrsim \epsilon^{-1/2}$. If we want the enhanced dissipation to dominate we then need

$$\tau_{ED} \approx \nu^{-1/3} \ll \epsilon^{-1/2} \lesssim \tau_{NL}.$$

Open problems

- No-penetration boundaries? (Bouchet-Morita and C. Zillinger)

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- No-penetration boundaries? (Bouchet-Morita and C. Zillinger)
- Other shear flows? (linear decay by Hao Jia-Sverak)
- Damping asymmetries in radial vortices?
- Do any of these ideas apply to Vlasov ? (with Bedrossian and Mouhot): New proof of the Landau damping of (Mouhot-Villani)

Thank you for your attention!