# The isospectral torus of quasi-periodic Schrödinger operators via periodic approximations

David Damanik[1], Michael Goldstein[2], and Milivoje Lukic[*2]

[1]Rice University – 6100 Main Street, Houston, Texas 77251-1892, United States
[2]University of Toronto – 40 St. George St. - Toronto, Ontario - M5S 2E4, Canada

**Abstract**

We study quasi-periodic Schrödinger operators $H = -\frac{d^2}{dx^2} + V$ in the regime of analytic sampling function and small coupling. More precisely, the potential is

$$V(x) = \sum_{m \in \mathbb{Z}^\nu} c(m) \exp(2\pi i m\omega x)$$

with $|c(m)| \leq \epsilon \exp(-\kappa|m|)$. Our main result is that any reflectionless potential $Q$ isospectral with $V$ is also quasi-periodic and in the same regime, with the same Diophantine frequency $\omega$, i.e.

$$Q(x) = \sum_{m \in \mathbb{Z}^\nu} d(m) \exp(2\pi i m\omega x)$$

with $|d(m)| \leq \sqrt{4\epsilon} \exp(-\frac{\kappa}{4}|m|)$. The proof relies on approximation by periodic potentials $\tilde{V}$, which are obtained by replacing the frequency $\omega$ by rational approximants $\tilde{\omega}$. We adapt the multiscale analysis, developed by Damanik–Goldstein for $V$, so that it applies to the periodic approximants $\tilde{V}$. This allows us to establish estimates for gap lengths and distances and Fourier coefficients of $\tilde{V}$ which are independent of period, unlike the standard estimates known in the theory of periodic Schrödinger operators. Starting from these estimates, we obtain the main result by comparing the isospectral tori and translation flows of $\tilde{V}$ and $V$.

[*]Speaker