Adaptive time-splitting methods for nonlinear Schrödinger equations in the semiclassical regime

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Abstract

We consider nonlinear Schrödinger equations involving a small parameter $\epsilon > 0$, a quadratic potential U, and a cubic nonlinearity,

$$i\partial_t \psi(t) = -\epsilon\,\Delta\psi(t) + \tfrac{1}{\epsilon}(U + |\psi(t)|^2)\psi(t),$$

$$\psi(0) = \psi_0,$$

and study the error behavior of time-splitting methods, extending the work from Descombes and Thalhammer (2012).
By suitable integral representations for the local error $\mathcal{L}(t)$ of the Lie–Trotter and Strang splitting methods, we deduce estimates that reflect the dependence on the time step t and the parameter ϵ,

$$\|\mathcal{L}_{Lie}(t)\|_{L^2} \le C\big(t^2 + t^3(\tfrac{1}{\epsilon} + \epsilon) + t^4(\tfrac{1}{\epsilon^2} + 1)\big) + t^2\mathcal{O}\big((\tfrac{t}{\epsilon})^3\big)\big(1 + \mathcal{O}(\varepsilon^2)\big),$$

$$\|\mathcal{L}_{Strang}(t)\|_{L^2} \le C\big(t^3(\tfrac{1}{\epsilon} + \epsilon) + t^4(\tfrac{1}{\epsilon^2} + 1)\big) + t^2\mathcal{O}\big((\tfrac{t}{\epsilon})^3\big)\big(1 + \mathcal{O}(\varepsilon^2)\big).$$

Numerical examples for the regimes $t > \epsilon$ and $t < \epsilon$ confirm these bounds. We also introduce a posteriori local error estimators and illustrate their performance, in particular for adaptive choice of the time steps.

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