On local and nonlocal variational constants of motion

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Abstract

Let \( q(t) \) be a solution to Euler-Lagrange equation for a smooth Lagrangian \( L(t, q, \dot{q}) \), with \( q \) in an open set of \( \mathbb{R}^n \), and let \( q_\lambda(t) \), \( \lambda \in \mathbb{R} \), be a smooth family of perturbed motions, such that \( q_0(t) \equiv q(t) \). Then the following function is constant:

\[
 t \mapsto \partial_\lambda L(t, q_\lambda(t), \dot{q}_\lambda(t)) \cdot \partial_\lambda q_\lambda(t) \bigg|_{\lambda=0} - \int_{t_0}^t \frac{\partial}{\partial \lambda} L(s, q_\lambda(s), \dot{q}_\lambda(s)) \bigg|_{\lambda=0} ds
\]

(\( \partial_\lambda \) gradient with respect to the vector \( \dot{q} \) and \( \cdot \) scalar product in \( \mathbb{R}^n \)). This constant of motion is generally nonlocal and trivial.

We can get genuine first integrals as for \( L = \frac{1}{2} \| \dot{q} \|^2 - U(q) \) with \( U \) homogeneous of degree \(-2\), in particular Calogero’s potential, and \( q_\lambda(t) = e^{\lambda} q(e^{-2\lambda}t) \). This example is taken from:


We also find nonlocal constants of motion which give global existence and asymptotic estimates for the solutions of \( \ddot{q} = -k\dot{q} - \partial_q U(q) \), when \( k > 0 \) and \( U : \mathbb{R}^n \to \mathbb{R} \) is bounded from below.

Finally, we show a nonlocal constant of motion for the Maxwell-Bloch system in Caşu’s Lagrangian formulation which leads to separation of one of the variables.