On local and nonlocal variational constants of motion

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Abstract

Let $q(t)$ be a solution to Euler-Lagrange equation for a smooth Lagrangian $L(t, q, \dot{q})$, with q in an open set of R^n, and let $q_\lambda(t)$, $\lambda \in R$, be a smooth family of perturbed motions, such that $q_0(t) \equiv q(t)$. Then the following function is constant:

$$t \mapsto \partial_{\dot{q}} L\big(t, q(t), \dot{q}(t)\big) \cdot \partial_\lambda q_\lambda(t)\big|_{\lambda=0} - \int_{t_0}^{t} \frac{\partial}{\partial \lambda} L\big(s, q_\lambda(s), \dot{q}_\lambda(s)\big)\Big|_{\lambda=0} ds$$

($\partial_{\dot{q}}$ gradient with respect to the vector $\dot{q}$ and $\cdot$ scalar product in R^n). This constant of motion is generally *nonlocal* and trivial.
We can get genuine first integrals as for $L = \frac{1}{2}\|\dot{q}\|^2 - U(q)$ with U homogeneous of degree -2, in particular Calogero's potential, and $q_\lambda(t) = e^\lambda\, q\big(e^{-2\lambda} t\big)$. This example is taken from: G. Gorni, G. Zampieri. Revisiting Noether's theorem on constants of motion. Journal of Nonlinear Mathematical Physics 21 (2014), 43-73.
We also find nonlocal constants of motion which give global existence and asymptotic estimates for the solutions of $\ddot{q} = -k\dot{q} - \partial_q U(q)$, when $k > 0$ and $U : R^n \to R$ is bounded from below.

Finally, we show a nonlocal constant of motion for the Maxwell-Bloch system in Caşu's Lagrangian formulation which leads to separation of one of the variables.

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