Some weighted estimates for elliptic operators with singular data

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Abstract

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $n \geq 2$, not necessarily bounded or regular. Consider in $\Omega$ the elliptic operator

$$L = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i} + a,$$

where the coefficients $a_{ij}$ are locally VMO while the data $a_i, a$ belong to a weighted Sobolev space whose weight $\sigma$ is a function of distance type from a nonempty subset of $\partial \Omega$. We prove that if $s \in \mathbb{R}$, $p \in [n/2, +\infty[$ and $u$ is a solution of the problem

\begin{align*}
&\begin{cases}
  u \in W^{2,p}_{loc}(\Omega) \\
  Lu \geq f & f \in L^p_{loc}(\Omega) \\
  \limsup_{x \to x_0} \sigma^s(x) u(x) \leq 0 & \forall x_0 \in \partial \Omega \\
  \limsup_{|x| \to +\infty} \sigma^s(x) u(x) \leq 0 & \text{if } \Omega \text{ is unbounded,}
\end{cases}
\end{align*}

then $u$ verifies the following Aleksandrov type estimate

$$\sup_{x \in \Omega} \sigma^s(x) u(x) \leq c \left( \frac{1}{|B|} \int_B |\sigma^{s+2} f^-|^p \, dx \right)^{\frac{1}{p}},$$

where $B \subset \subset \Omega$ is an open ball, $f^-$ is the negative part of $f$ and $c \in \mathbb{R}_+$ is independent of $u$. As a consequence, some uniqueness results for singular elliptic problems are obtained.