EquaDiff 2015

Lyon, France

July 6-10, 2015

# Toward a Smooth Ergodic Theory for Infinite Dimensional Systems

Lai-Sang Young

Courant Institute, NYU

http://www.cims.nyu.edu/~lsy/

- For dynamical systems defined by maps or flows (ODEs),
  *ergodic theory* offers a description of their global dynamics
  in terms of averages & almost sure behaviors
  (useful for complicated dynamics, less so for special orbits)

- In finite dim, there is a fairly well developed *smooth* ergodic theory.
  This talk is about : extension of this theory to infinite dimensions.

  * sample results    * differences between fin and infinite dims

- Basic objects in this theory are (1) *phase space* $X$ ,
  (2) *dynamics* $f^t$ , and (3) notion of what is *typical* $\mu$

- Outline of this talk

  I. Dynamical setting for certain classes of PDEs
  II. 3 sample results for general $(X, f^t, \mu)$
  III. Existence vs observability of dynamical complexity

- For dynamical systems defined by maps or flows (ODEs), *ergodic theory* offers a description of their global dynamics in terms of averages & almost sure behaviors

  (useful for complicated dynamics, less so for special orbits)

- In finite dim, there is a fairly well developed *smooth* ergodic theory.

  This talk is about : extension of this theory to infinite dimensions.

  * sample results  * differences between fin and infinite dims

- Basic objects in this theory are (1) *phase space* $X$ , (2) *dynamics* $f^t$ , and (3) notion of what is *typical* $\mu$

- <u>Outline of this talk</u>

  I. Dynamical setting for certain classes of PDEs

  II. 3 sample results for general $(X, f^t, \mu)$

  III. Existence vs observability of dynamical complexity

- For dynamical systems defined by maps or flows (ODEs),
  *ergodic theory* offers a description of their global dynamics
  in terms of averages & almost sure behaviors
  (useful for complicated dynamics, less so for special orbits)

- In finite dim, there is a fairly well developed *smooth* ergodic theory.
  This talk is about : extension of this theory to infinite dimensions.
  * sample results   * differences between fin and infinite dims

- Basic objects in this theory are (1) *phase space* $X$ ,
  (2) *dynamics* $f^t$ , and (3) notion of what is *typical* $\mu$

- Outline of this talk

I. Dynamical setting for certain classes of PDEs
II. 3 sample results for general $(X, f^t, \mu)$
III. Existence vs observability of dynamical complexity

- For dynamical systems defined by maps or flows (ODEs),
  *ergodic theory* offers a description of their global dynamics
  in terms of averages & almost sure behaviors
  (useful for complicated dynamics, less so for special orbits)

- In finite dim, there is a fairly well developed *smooth* ergodic theory.
  This talk is about : extension of this theory to infinite dimensions.
  * sample results * differences between fin and infinite dims

- Basic objects in this theory are (1) *phase space* $X$ ,
  (2) *dynamics* $f^t$ , and (3) notion of what is *typical* $\mu$

- <u>Outline of this talk</u>

  I. Dynamical setting for certain classes of PDEs
  II. 3 sample results for general $(X, f^t, \mu)$
  III. Existence vs observability of dynamical complexity

# I. Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au = f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a $C^r$ dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in $X$ for all $t \geq 0$ ,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each $t$

This imposes restriction on the choice of $(X, \|\cdot\|)$

**Remark** : (1) is necessary for purposes of studying global dynamics.
(3) is important if one is to leverage finite dim geom/differentiable techniques
[Can do with less for e.g. construction of special solutions etc.]

# I. Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au = f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a $C^r$ dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in $X$ for all $t \geq 0$ ,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each $t$

This imposes restriction on the choice of $(X, \|\cdot\|)$

**Remark** : (1) is necessary for purposes of studying global dynamics.
(3) is important if one is to leverage finite dim geom/differentiable techniques
[Can do with less for e.g. construction of special solutions etc.]

# I. Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au \; = \; f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a $C^r$ dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in $X$ for all $t \geq 0$ ,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each $t$

This imposes restriction on the choice of $(X, \|\cdot\|)$

**Remark** : (1) is necessary for purposes of studying global dynamics.
(3) is important if one is to leverage finite dim geom/differentiable techniques
[Can do with less for e.g. construction of special solutions etc.]

# I. Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au = f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a $C^r$ dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in $X$ for all $t \geq 0$ ,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each $t$

This imposes restriction on the choice of $(X, \|\cdot\|)$

**Remark** : (1) is necessary for purposes of studying global dynamics.

(3) is important if one is to leverage finite dim geom/differentiable techniques

[Can do with less for e.g. construction of special solutions etc.]

# I. Dynamical setting for certain classes of PDEs

Consider $$\frac{du}{dt} + Au = f(u)$$

where $u \in X =$ function space, $A =$ linear operator, $f =$ nonlinear term

To define a $C^r$ dynamical system, need $(X, \|\cdot\|)$ s.t.

(1) $u_0 \in X \implies u(t)$ exists and is unique in $X$ for all $t \geq 0$ ,

so semiflow $f^t : X \to X$ is well defined

(2) $t \mapsto u(t)$ is continuous for $t \geq 0$

(3) $f^t \in C^r$ for each $t$

This imposes restriction on the choice of $(X, \|\cdot\|)$

**Remark** : (1) is necessary for purposes of studying global dynamics.

(3) is important if one is to leverage finite dim geom/differentiable techniques

[Can do with less for e.g. construction of special solutions etc.]

## A sample result (Henry ~ 1980)

$X =$ Banach sp , $A =$ sectorial operator (equiv $e^{-At}$ analytic semigp)

Fact : There exist $\{(X^\alpha, \|\cdot\|_\alpha)\}$ interpolation spaces

$\alpha \longleftarrow$

$X^1 = \mathcal{D}(A)$ $\qquad X^\alpha \qquad$ $X^0 = X$

stronger $\longleftarrow$ $\|\cdot\|_\alpha$ $\longrightarrow$ weaker

**THEOREM.** Given

$$\partial_t u + Au = F(u) \ , \qquad u \in X, \quad A \ \text{sectorial} \ ,$$

*if* for some $\alpha \in [0,1)$ , $F : X^\alpha \to X$ is $C^r, r \geq 1$ ,

*then* for all $u_0 \in X^\alpha$ , there exists a unique $u(t) \in X^\alpha$ and $f^t : u_0 \mapsto u(t)$ is a $C^r$ map from $(X^\alpha, \|\cdot\|_\alpha)$ to $(X^\alpha, \|\cdot\|_\alpha)$ .

Solution here means *mild solution*, i.e.

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} F(u(s))ds$$

# A sample result (Henry ~ 1980)

$X =$ Banach sp , $A =$ sectorial operator (equiv $e^{-At}$ analytic semigp)

Fact : There exist $\{(X^\alpha, \|\cdot\|_\alpha)\}$ interpolation spaces

$\alpha$

$X^1 = \mathcal{D}(A)$ $\qquad X^\alpha \qquad$ $X^0 = X$

stronger $\longleftarrow$ $\|\cdot\|_\alpha$ $\longrightarrow$ weaker

**THEOREM.** Given

$$\partial_t u + Au = F(u) \ , \qquad u \in X, \quad A \ \text{sectorial} \ ,$$

*if* for some $\alpha \in [0,1)$ , $F : X^\alpha \to X$ is $C^r, r \geq 1$ ,

*then* for all $u_0 \in X^\alpha$ , there exists a unique $u(t) \in X^\alpha$
and $f^t : u_0 \mapsto u(t)$ is a $C^r$ map from $(X^\alpha, \|\cdot\|_\alpha)$ to $(X^\alpha, \|\cdot\|_\alpha)$ .

Solution here means *mild solution*, i.e.

$$u(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} F(u(s)) ds$$

# A sample result (Henry ~ 1980)

$X =$ Banach sp , $A =$ sectorial operator (equiv $e^{-At}$ analytic semigp)

Fact : There exist $\{(X^\alpha, \|\cdot\|_\alpha)\}$ interpolation spaces

$\alpha$

$X^1 = \mathcal{D}(A)$ $\quad X^\alpha \quad$ $X^0 = X$

stronger $\longleftarrow$ $\|\cdot\|_\alpha$ $\longrightarrow$ weaker

**THEOREM.** Given

$$\partial_t u + Au = F(u) \;, \qquad u \in X, \quad A \text{ sectorial ,}$$

*if* for some $\alpha \in [0,1)$ , $F : X^\alpha \to X$ is $C^r, r \geq 1$ ,

*then* for all $u_0 \in X^\alpha$ , there exists a unique $u(t) \in X^\alpha$

and $f^t : u_0 \mapsto u(t)$ is a $C^r$ map from $(X^\alpha, \|\cdot\|_\alpha)$ to $(X^\alpha, \|\cdot\|_\alpha)$ .

Solution here means *mild solution*, i.e.

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s))ds$$

## II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0,\infty) \times X \to X$ cts semiflow , $f^t(x) = F(t, x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

### Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e. x$ , $T_xM = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

# II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0,\infty) \times X \to X$ cts semiflow , $f^t(x) = F(t,x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

## Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e.x$ , $T_xM = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim\limits_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

# II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0, \infty) \times X \to X$ cts semiflow , $f^t(x) = F(t, x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

## Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e. x$ , $T_xM = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

# II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0,\infty) \times X \to X$ cts semiflow , $f^t(x) = F(t,x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

## Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e.x$ , $T_xM = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

# II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0, \infty) \times X \to X$ cts semiflow , $f^t(x) = F(t, x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

## Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e.x$ , $T_xM = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim\limits_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

# II. Three sample results for general $(X, f, \mu)$

**Setting** : $X =$ Banach or Hilbert space

$F : [0, \infty) \times X \to X$ cts semiflow , $f^t(x) = F(t, x)$

*Assume* (1) $F|_{(0,\infty)\times X}$ is $C^2$

(2) $f^t$ , $Df^t_x$ injective [backward uniqueness]

(3) existence of compact $A \subset X$ , $f^t(A) = A$ [attractor]

*Basic fact* : existence of invariant prob measures on $A$ (often many)

## Result # 1: Lyapunov exponents (Multiplicative Ergodic Theorem)

**THEOREM** (finite dim) (Oseledec ~ 68) : $f : M^d \to M^d$ diffeo, $\mu$ inv prob

(Ergodic version) There exist $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ s.t.

at $\mu - a.e.x$ , $T_x M = E_1(x) \oplus \cdots \oplus E_r(x)$

and for all $v \in E_i(x)$ , $\lim_{n\to\pm\infty} \frac{1}{|n|} \log \|Df^n_x(v)\| = \pm\lambda_i$

$\angle(E_i, E_j)$ varying slowly along orbits

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible

2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n)$$ well defined $\mu - a.e.$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist

$\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , Lyap exp $= \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \leq \log \kappa'$ [angles repl by $\|\pi^{E_i, F}\|$ proj along complements]

Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible

2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0}\{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n)$$ well defined $\mu - a.e.$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist

$\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , Lyap exp $= \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \leq \log \kappa'$

[angles repl by $\|\pi^{E_i,F}\|$ proj along complements]

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible
2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n) \quad \text{well defined} \quad \mu - a.e.$$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist

$\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , Lyap exp $= \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \le \log \kappa'$

[angles repl by $\|\pi^{E_i,F}\|$ proj along complements]

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible
2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n) \quad \text{well defined} \quad \mu - a.e.$$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist $\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , Lyap exp $= \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \le \log \kappa'$

[angles repl by $\|\pi^{E_i,F}\|$
proj along complements]

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible
2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite } \# \text{ of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n)$$ well defined $\mu - a.e.$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist

$\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , Lyap exp $= \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \leq \log \kappa'$ [angles repl by $\|\pi^{E_i, F}\|$ proj along complements]

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible
2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n)$$ well defined $\mu - a.e.$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist $\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , $\text{Lyap exp } = \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \leq \log \kappa'$

[angles repl by $\|\pi^{E_i,F}\|$ proj along complements]

## Biggest differences between finite and infinite dims :

1. $Df_x$ not invertible

2. Presence of essential spectrum

For single operator $T \in \mathcal{L}(X, X)$ , define

$$\kappa_o(T) = \inf_{r>0} \{T(B_1) \text{ can be covered by finite \# of } B_r\}$$

Kuratowski measure of noncpctness (essential spectral radius)

In ergodic theory setting,

$$\log \kappa(x) = \lim_{n\to\infty} \frac{1}{n} \log \kappa_0(Df_x^n) \quad \text{well defined} \quad \mu - a.e.$$

**THEOREM** (Ruelle, Mane, Thieullen 80s, Lian-Lu 2010, . . . )

(Ergodic version) In inf dim setting above, for any $\kappa' > \kappa$ , there exist $\lambda_1 > \cdots > \lambda_r > \kappa'$ and a decomp $T_x = \oplus_{i=1}^r E_i(x) \oplus F(x)$

s.t. $\dim(E_i) < \infty$ , $Df_x(E_i(x)) = E_i(f(x))$ , $\text{Lyap exp} = \lambda_i$

$\dim(F) = \infty$ , $Df_x(F(x)) \subset F(f(x))$

and $\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| \leq \log \kappa'$

[angles repl by $\|\pi^{E_i,F}\|$ proj along complements]

Remark : To distinguish *expanding* , *neutral* & *contracting* directions, i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on

Also known : local stable and unstable manifolds $W^s, W^u \; \mu - a.e.$

# Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0 \; \forall \, i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either
$\mu$ is supp on an unstable periodic periodic solution
or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow \lambda_i > 0$ for some $i$ ],
then (i) almost all entropy is supported on horseshoes
(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Remark : To distinguish *expanding* , *neutral* & *contracting* directions, i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on

Also known : local stable and unstable manifolds $W^s, \; W^u \;\; \mu - a.e.$

## Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0 \; \forall \, i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either

$\mu$ is supp on an unstable periodic periodic solution

or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow \lambda_i > 0$ for some $i$ ],

then (i) almost all entropy is supported on horseshoes

(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Remark : To distinguish *expanding* , *neutral* & *contracting* directions,
i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on

Also known : local stable and unstable manifolds $W^s, \; W^u \; \mu - a.e.$

## Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0 \; \forall \, i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either
$\mu$ is supp on an unstable periodic periodic solution
or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow \lambda_i > 0$ for some $i$ ],
then (i) almost all entropy is supported on horseshoes
(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Remark : To distinguish *expanding* , *neutral* & *contracting* directions, i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on

Also known : local stable and unstable manifolds $W^s, \ W^u \ \mu - a.e.$

## Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0 \ \forall \ i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either

$\mu$ is supp on an unstable periodic periodic solution

or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow \lambda_i > 0$ for some $i$ ],

then (i) almost all entropy is supported on horseshoes

(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Remark : To distinguish *expanding* , *neutral* & *contracting* directions,
i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on
Also known : local stable and unstable manifolds $W^s, W^u$ $\mu - a.e.$

## Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0 \; \forall \, i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either
$\mu$ is supp on an unstable periodic periodic solution
or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow$ $\lambda_i > 0$ for some $i$ ],
then (i) almost all entropy is supported on horseshoes
(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Remark : To distinguish *expanding* , *neutral* & *contracting* directions,
i.e. to have $E^u \oplus E^c \oplus E^s$ , assume $\log \kappa < 0$ from here on
Also known : local stable and unstable manifolds $W^s,\ W^u\ \mu - a.e.$

## Result #2. Lyap exp, periodic solutions & horseshoes

$X =$ Hilbert space, $f^t = C^2$ semi flow , $\lambda_i =$ Lyap exp

Finite dim diffeo version of following proved by Katok (1980)

**THEOREM** (Lian-Young 2013)

*Assume* (a) ergodic and (b) exactly one 0-exp (flow direction). *Then :*

(1) If $\lambda_i < 0\ \forall\ i$ , then $\mu$ is supported on a stable periodic solution

(2) If $\lambda_i > 0$ for some $i$ , then either
$\mu$ is supp on an unstable periodic periodic solution
or there are infinitely many unstable periodic solutions

(3) If $h_\mu(f) > 0$ (metric entropy) [which $\Longrightarrow$ $\lambda_i > 0$ for some $i$ ],
then (i) almost all entropy is supported on horseshoes
(ii) # periodic solutions of period $\leq s$ grows exp with $s$

Review

# Entropy (Kolmogorov-Sinai 1959)

measure of dynamical complexity in the sense of information theory

$\alpha = \{A_1, \cdots, A_k\}$ partition , $H(\alpha) = -\sum p_i \log p_i$ , $p_i = \mu(A_i)$

Then
$$h_\mu(f) = \sup_\alpha H\left(\alpha \mid \bigvee_1^\infty f^{-i}\alpha\right)$$

**Interpretation** : amount of uncertainty in predicting $\alpha$- location of a point given its past (or future)

# Horseshoes (Smale 1960s finite dim)




Dynamical complexity :
existence of orbits corresp to all sequences in $\{L, R\}^{\mathbb{Z}}$

**Interpretation in inf dim :** existence of two (distinguishable) sets of functions (profiles) $\mathcal{U}_0, \mathcal{U}_1$ s.t.
given any sequence $(a_i), a_i = 0$ or $1$ ,
there is a solution $u(t)$ s.t.
$u(it_0) \in \mathcal{U}_{a_i}$ for some $t_0$.

Review

# Entropy (Kolmogorov-Sinai 1959)

measure of dynamical complexity in the sense of information theory

$\alpha = \{A_1, \cdots, A_k\}$ partition , $H(\alpha) = -\sum p_i \log p_i$ , $p_i = \mu(A_i)$

Then $$h_\mu(f) = \sup_\alpha H\left(\alpha \mid \bigvee_1^\infty f^{-i}\alpha\right)$$

**Interpretation** : amount of uncertainty in predicting $\alpha$- location of a point given its past (or future)

# Horseshoes (Smale 1960s finite dim)




Dynamical complexity :
existence of orbits corresp to all sequences in $\{L, R\}^{\mathbb{Z}}$

**Interpretation in inf dim :** existence of two (distinguishable) sets of functions (profiles) $\mathcal{U}_0, \mathcal{U}_1$ s.t.
given any sequence $(a_i), a_i = 0$ or $1$ ,
there is a solution $u(t)$ s.t.
$u(it_0) \in \mathcal{U}_{a_i}$ for some $t_0$.

Review

# Entropy (Kolmogorov-Sinai 1959)

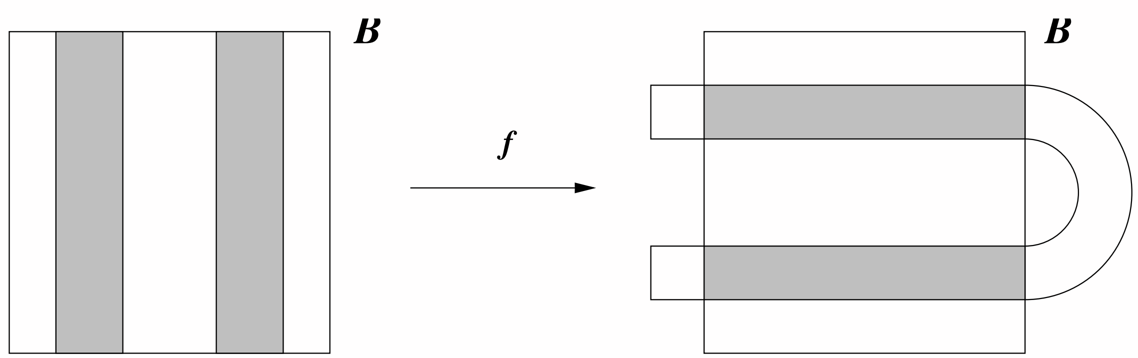
measure of dynamical complexity in the sense of information theory

$\alpha = \{A_1, \cdots, A_k\}$ partition , $H(\alpha) = -\sum p_i \log p_i$ , $p_i = \mu(A_i)$

Then
$$h_\mu(f) = \sup_\alpha H\left(\alpha \mid \bigvee_1^\infty f^{-i}\alpha\right)$$

**Interpretation** : amount of uncertainty in predicting $\alpha$- location of a point given its past (or future)

# Horseshoes (Smale 1960s finite dim)




Dynamical complexity :
existence of orbits corresp to all sequences in $\{L, R\}^{\mathbb{Z}}$

**Interpretation in inf dim :** existence of two (distinguishable) sets of functions (profiles) $\mathcal{U}_0, \mathcal{U}_1$ s.t.
given any sequence $(a_i), a_i = 0 \text{ or } 1$ ,
there is a solution $u(t)$ s.t.
$u(it_0) \in \mathcal{U}_{a_i}$ for some $t_0$.

## Entropy (Kolmogorov-Sinai 1959)

Review

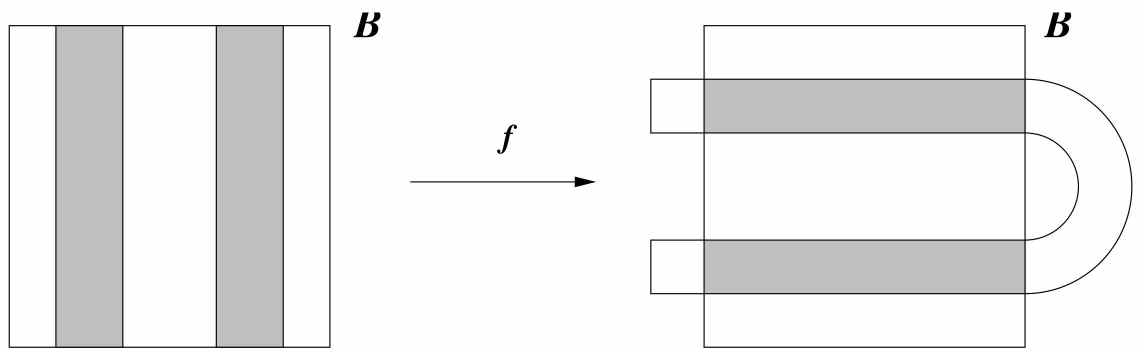
measure of dynamical complexity in the sense of information theory

$\alpha = \{A_1, \cdots, A_k\}$ partition , $H(\alpha) = -\sum p_i \log p_i$ , $p_i = \mu(A_i)$

Then $$h_\mu(f) = \sup_\alpha H\left(\alpha \mid \bigvee_1^\infty f^{-i}\alpha\right)$$

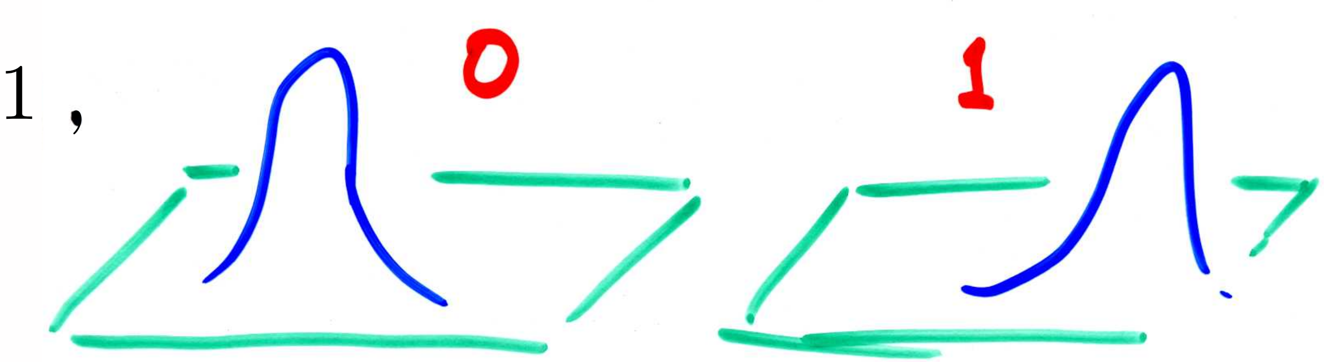
**Interpretation** : amount of uncertainty in predicting $\alpha$- location of a point given its past (or future)

## Horseshoes (Smale 1960s finite dim)




Dynamical complexity :
existence of orbits corresp to all sequences in $\{L, R\}^{\mathbb{Z}}$

**Interpretation in inf dim :** existence of two (distinguishable) sets of functions (profiles) $\mathcal{U}_0, \mathcal{U}_1$ s.t.
given any sequence $(a_i), a_i = 0 \text{ or } 1$ ,
there is a solution $u(t)$ s.t.
$u(it_0) \in \mathcal{U}_{a_i}$ for some $t_0$ .



# Result #3. Entropy, Lyap exp and SRB measures

Setting as before : $f$ is $C^2$ map of Banach space etc.

**THEOREM** (Thieullen 1980s) For any invariant measure $\mu$ ,

$$h_\mu(f) \le \int \sum_i \lambda_i^+ \dim E_i \; d\mu$$

This generalizes Ruelle's Inequality first proved in finite dim.

**THEOREM** (Blumenthal-Young 2015) Assume no 0 Lyap exponents.
Then $\mu$ is an SRB measure if and only if $h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu$ .

This generalizes results of Ledrappier and Ledr-Strelcyn for fin dim diffeos.

**Definition.** $\mu$ is called an *SRB measure* if $(f, \mu)$ has pos Lyap exp
and $\mu$ has smooth conditional densities on unstable manifolds.

**Interpretation :**

Inequality says entropy dominated by exp rate of divergence of solutions.
Equality says the divergence rates = entropy iff measure in "Lebesgue" meas class
in unstable directions

# Result #3. Entropy, Lyap exp and SRB measures

Setting as before : $f$ is $C^2$ map of Banach space etc.

**THEOREM** (Thieullen 1980s) For any invariant measure $\mu$ ,

$$h_\mu(f) \le \int \sum_i \lambda_i^+ \dim E_i \; d\mu$$

This generalizes Ruelle's Inequality first proved in finite dim.

**THEOREM** (Blumenthal-Young 2015) Assume no 0 Lyap exponents. Then $\mu$ is an SRB measure if and only if $h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu$ .

This generalizes results of Ledrappier and Ledr-Strelcyn for fin dim diffeos.

**Definition.** $\mu$ is called an *SRB measure* if $(f, \mu)$ has pos Lyap exp and $\mu$ has smooth conditional densities on unstable manifolds.

**Interpretation :**

Inequality says entropy dominated by exp rate of divergence of solutions.
Equality says the divergence rates = entropy iff measure in "Lebesgue" meas class in unstable directions

# Result #3. Entropy, Lyap exp and SRB measures

Setting as before : $f$ is $C^2$ map of Banach space etc.

**THEOREM** (Thieullen 1980s) For any invariant measure $\mu$ ,

$$h_\mu(f) \le \int \sum_i \lambda_i^+ \dim E_i \, d\mu$$

This generalizes Ruelle's Inequality first proved in finite dim.

**THEOREM** (Blumenthal-Young 2015) Assume no 0 Lyap exponents. Then $\mu$ is an SRB measure if and only if $h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu$ .

This generalizes results of Ledrappier and Ledr-Strelcyn for fin dim diffeos.

**Definition.** $\mu$ is called an *SRB measure* if $(f, \mu)$ has pos Lyap exp and $\mu$ has smooth conditional densities on unstable manifolds.

**Interpretation :**

Inequality says entropy dominated by exp rate of divergence of solutions.
Equality says the divergence rates = entropy iff measure in "Lebesgue" meas class in unstable directions

# Result #3. Entropy, Lyap exp and SRB measures

Setting as before : $f$ is $C^2$ map of Banach space etc.

**THEOREM** (Thieullen 1980s) For any invariant measure $\mu$ ,

$$h_\mu(f) \le \int \sum_i \lambda_i^+ \dim E_i \, d\mu$$

This generalizes Ruelle's Inequality first proved in finite dim.

**THEOREM** (Blumenthal-Young 2015) Assume no 0 Lyap exponents. Then $\mu$ is an SRB measure if and only if $h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu$ .

This generalizes results of Ledrappier and Ledr-Strelcyn for fin dim diffeos.

**Definition.** $\mu$ is called an *SRB measure* if $(f, \mu)$ has pos Lyap exp and $\mu$ has smooth conditional densities on unstable manifolds.

**Interpretation :**

Inequality says entropy dominated by exp rate of divergence of solutions.

Equality says the divergence rates = entropy iff measure in "Lebesgue" meas class in unstable directions

# Result #3. Entropy, Lyap exp and SRB measures

Setting as before : $f$ is $C^2$ map of Banach space etc.

**THEOREM** (Thieullen 1980s) For any invariant measure $\mu$ ,

$$h_\mu(f) \leq \int \sum_i \lambda_i^+ \dim E_i \, d\mu$$

This generalizes Ruelle's Inequality first proved in finite dim.

**THEOREM** (Blumenthal-Young 2015) Assume no 0 Lyap exponents.
Then $\mu$ is an SRB measure if and only if $h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu$ .

This generalizes results of Ledrappier and Ledr-Strelcyn for fin dim diffeos.

**Definition.** $\mu$ is called an *SRB measure* if $(f, \mu)$ has pos Lyap exp and $\mu$ has smooth conditional densities on unstable manifolds.

**Interpretation :**

Inequality says entropy dominated by exp rate of divergence of solutions.
Equality says the divergence rates = entropy iff measure in "Lebesgue" meas class in unstable directions

How to make sense of

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u$$

- $h \sim$ growth rate of # $\mu$–typical distinguishable $n$-orbits

  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$ , a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$\boxed{h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u}$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits

  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$, a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$\boxed{h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u}$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits
  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$, a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k = \text{ Leb } (B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits

  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$ , a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$\boxed{h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u}$$

- $h \sim$ growth rate of # $\mu$–typical distinguishable $n$-orbits
  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$, a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$\boxed{h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \; d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u}$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits
  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

## *Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$, a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits

  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

  Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

## *Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$ , a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$\boxed{h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu \qquad \text{iff} \qquad \mu \sim \ \text{Leb on } W^u}$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits
  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$ , a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

How to make sense of

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim E_i \, d\mu \qquad \text{iff} \qquad \mu \sim \text{ Leb on } W^u$$

- $h \sim$ growth rate of # $\mu-$typical distinguishable $n$-orbits

  $\sim$ - rate at which typical $\mu\left(\bigvee_0^n (f^{-i}\alpha)(x)\right)$ decreases

Shannon-McMillan -Breiman Theorem

- $\sum_i \lambda_i^+ \dim E_i \sim$ rate of volume growth in unstable directions
- hence 2 growth rates are equal when $\mu \sim$ volume.

*Volumes on Banach spaces ???*

First, note $\dim(E^u) < \infty$ in dissipative systems.

Can define, on each subspace $E \subset X$, $\dim E = k$ , a vol element $m_E$

s.t. $m_E(B_1) = c_k$ where $B_1 =$ unit ball, $c_k =$ Leb $(B_1)$ in $\mathbb{R}^k$

(Busemann-Hausdorff vol in Finsler geom)

*Regularity issues* : how $m_E$ and assoc $\det(df_x|E)$ vary with $E$

We proved : sufficient for results above (but can be problematic elsewhere)

Other technical issues : e.g. regarding dynamical objects characterized by "backward iterates" + their regularity

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits
  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

$$\boxed{\text{observable events = positive Leb meas sets}}$$

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic, no 0-Lyap exp,

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x \qquad \text{for all cts } \varphi$$

Follows from the *absolute continuity of $W^s$ foliation.* (Pugh-Shub '90)

## Infinite dim counterpart ?

e.g. pos meas set of Fourier coefficients ?
probing phase space with finite dim spaces?

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits
  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

$$\boxed{\text{observable events = positive Leb meas sets}}$$

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic, no 0-Lyap exp,

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x \qquad \text{for all cts } \varphi$$

Follows from the *absolute continuity of $W^s$ foliation.* (Pugh-Shub '90)

## Infinite dim counterpart ?

e.g. pos meas set of Fourier coefficients ?
probing phase space with finite dim spaces?

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits
  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

$$\boxed{\text{observable events = positive Leb meas sets}}$$

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic, no 0-Lyap exp,

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x \qquad \text{for all cts } \varphi$$

Follows from the *absolute continuity of* $W^s$ *foliation.* (Pugh-Shub '90)

## Infinite dim counterpart ?

e.g. pos meas set of Fourier coefficients ?
probing phase space with finite dim spaces?

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits
  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

observable events = positive Leb meas sets

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic SRB, no 0-Lyap exp,

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x \qquad \text{for all cts } \varphi$$

Follows from the *absolute continuity of* $W^s$ *foliation.* (Pugh-Shub '90)

**Infinite dim counterpart ?**

e.g. pos meas set of Fourier coefficients ?
probing phase space with finite dim spaces?

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits

  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

observable events = positive Leb meas sets

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic SRB, no 0-Lyap exp,

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi\, d\mu \quad \text{Leb-a.e. } x \qquad \text{for all cts } \varphi$$

Follows from the *absolute continuity of* $W^s$ *foliation.* (Pugh-Shub '90)

**Infinite dim counterpart ?**

e.g. pos meas set of Fourier coefficients ?

probing phase space with finite dim spaces?

# III. Dynamical complexity: existence vs observability

- Presence of *horseshoes* implies *existence* of unstable orbits
  — almost all other initial conditions may tend to stable equilibrium

- In **finite dim**, a more persistent, observable kind of chaos/instability is *pos Lyap exp Leb-a.e.* or on pos Leb meas set, i.e.

observable events = positive Leb meas sets

- Hamiltonian systems : Liouville measure natural
- Dissipative systems : SRB measures are natural invariant measures

**THEOREM.** For $(f, \mu)$ ergodic SRB, no 0-Lyap exp

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi d\mu \quad \text{Leb-a.e. } x$$ for all cts $\varphi$

Follows from the *absolute continuity of* $W^s$ *foliation.* (Pugh-Shub '90)

## Infinite dim counterpart ?

e.g. pos meas set of Fourier coefficients ?
probing phase space with finite dim spaces?

# Sample result #1. Absolute continuity of stable foliations & a notion of ``almost everywhere" in Banach spaces

## Setting 1. Center / initial manifolds

Existence of $W^c$ proved many times
e.g. Constantin-Fois-Nicolaenko 80s, Chow, Sell, Mallet-Paret, Lu …..

Geometric conditions

$X$ = Banach space, $f : X \to X$ $C^{1+\alpha}$, $\alpha > 0$

(A1) Reference splitting $X = E^c \oplus E^s$ closed subspace, not nec invariant

(A2) Absorbing slab

$$\forall R, \exists R' \text{ s.t. } f(E^c \times B^s(R)) \subset E^c \times B^s(R')$$

(A3) Invariant cones

$\exists \mu \in (0,1), \lambda_c \in \mathbb{R}$ s.t. if $\mathcal{C}^c = \{v : \|\pi^s v\| \leq \mu \|\pi^c v\|\}$ ,

then $Df_x(\mathcal{C}^c(x)) \subset \mathcal{C}^c(fx)$ and $\forall v \in \mathcal{C}^c$ , $\|\pi^c Df_x v\| \geq e^\lambda \|\pi^c v\|$

and similar for $\mathcal{C}^s$, backward invariant, contract $\lambda_s < \min\{0, \lambda\}$

(A1)-(A3) satisfied by e.g.

$$u_t = \Delta u + g(u), \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0$$

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0, \ \gamma > 0 \ .$$

# Sample result #1. Absolute continuity of stable foliations & a notion of ``almost everywhere'' in Banach spaces

## Setting 1. Center / initial manifolds

Existence of $W^c$ proved many times e.g. Constantin-Fois-Nicolaenko 80s, Chow, Sell, Mallet-Paret, Lu …..

<u>Geometric conditions</u>

$X$ = Banach space, $f : X \to X$ $C^{1+\alpha}$, $\alpha > 0$

(A1) Reference splitting $X = E^c \oplus E^s$ closed subspace, not nec invariant

(A2) Absorbing slab

$$\forall R, \exists R' \text{ s.t. } f(E^c \times B^s(R)) \subset E^c \times B^s(R')$$

(A3) Invariant cones

$$\exists \mu \in (0,1), \lambda_c \in \mathbb{R} \quad \text{s.t. if} \quad \mathcal{C}^c = \{v : \|\pi^s v\| \leq \mu \|\pi^c v\|\} \ ,$$

$$\text{then} \quad Df_x(\mathcal{C}^c(x)) \subset \mathcal{C}^c(fx) \quad \text{and} \quad \forall v \in \mathcal{C}^c \ , \|\pi^c Df_x v\| \geq e^\lambda \|\pi^c v\|$$

and similar for $\mathcal{C}^s$, backward invariant, contract $\lambda_s < \min\{0, \lambda\}$

(A1)-(A3) satisfied by e.g.

$$u_t = \Delta u + g(u), \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0$$

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0, \ \gamma > 0 \ .$$

# Sample result #1. Absolute continuity of stable foliations & a notion of ``almost everywhere'' in Banach spaces

## Setting 1. Center / initial manifolds

Existence of $W^c$ proved many times e.g. Constantin-Fois-Nicolaenko 80s, Chow, Sell, Mallet-Paret, Lu …..

<u>Geometric conditions</u>

$X =$ Banach space, $f : X \to X$ $C^{1+\alpha}$, $\alpha > 0$

(A1) Reference splitting $X = E^c \oplus E^s$ closed subspace, not nec invariant

(A2) Absorbing slab

$$\forall R, \exists R' \text{ s.t. } f(E^c \times B^s(R)) \subset E^c \times B^s(R')$$

(A3) Invariant cones

$$\exists \mu \in (0,1), \lambda_c \in \mathbb{R} \quad \text{s.t. if} \quad \mathcal{C}^c = \{v : \|\pi^s v\| \le \mu \|\pi^c v\|\} \ ,$$

$$\text{then} \quad Df_x(\mathcal{C}^c(x)) \subset \mathcal{C}^c(fx) \quad \text{and} \quad \forall v \in \mathcal{C}^c \ , \|\pi^c Df_x v\| \ge e^{\lambda} \|\pi^c v\|$$

and similar for $\mathcal{C}^s$, backward invariant, contract $\lambda_s < \min\{0, \lambda\}$

(A1)-(A3) satisfied by e.g.

$$u_t = \Delta u + g(u), \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0$$

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0, \ \gamma > 0 \ .$$

# Sample result #1. Absolute continuity of stable foliations & a notion of ``almost everywhere'' in Banach spaces

## Setting 1. Center / initial manifolds

Existence of $W^c$ proved many times
e.g. Constantin-Fois-Nicolaenko 80s, Chow, Sell, Mallet-Paret, Lu …..

Geometric conditions

$X =$ Banach space, $f : X \to X \;\; C^{1+\alpha}, \;\; \alpha > 0$

(A1) Reference splitting $X = E^c \oplus E^s$ closed subspace, not nec invariant

(A2) Absorbing slab

$$\forall R, \exists R' \text{ s.t. } f(E^c \times B^s(R)) \subset E^c \times B^s(R')$$

(A3) Invariant cones

$$\exists \mu \in (0,1), \lambda_c \in \mathbb{R} \quad \text{s.t. if} \quad \mathcal{C}^c = \{v : \|\pi^s v\| \le \mu \|\pi^c v\|\} \; ,$$

$$\text{then} \quad Df_x(\mathcal{C}^c(x)) \subset \mathcal{C}^c(fx) \quad \text{and} \quad \forall v \in \mathcal{C}^c \; , \|\pi^c Df_x v\| \ge e^{\lambda} \|\pi^c v\|$$

and similar for $\mathcal{C}^s$, backward invariant, contract $\lambda_s < \min\{0, \lambda\}$

(A1)-(A3) satisfied by e.g.

$$u_t = \Delta u + g(u), \quad x \in \Omega \subset\subset \mathbb{R}^n, \; u|_{\partial\Omega} = 0$$

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n, \; u|_{\partial\Omega} = 0, \; \gamma > 0 \; .$$

# Sample result #1. Absolute continuity of stable foliations & a notion of ``almost everywhere'' in Banach spaces

## Setting 1. Center / initial manifolds

Existence of $W^c$ proved many times
e.g. Constantin-Fois-Nicolaenko 80s,
Chow, Sell, Mallet-Paret, Lu …..

Geometric conditions

$X =$ Banach space, $f : X \to X$ $C^{1+\alpha}$, $\alpha > 0$

(A1) Reference splitting $X = E^c \oplus E^s$ closed subspace, not nec invariant

(A2) Absorbing slab

$$\forall R, \exists R' \text{ s.t. } f(E^c \times B^s(R)) \subset E^c \times B^s(R')$$

(A3) Invariant cones

$$\exists \mu \in (0,1), \lambda_c \in \mathbb{R} \quad \text{s.t. if} \quad \mathcal{C}^c = \{v : \|\pi^s v\| \le \mu \|\pi^c v\|\} \ ,$$

$$\text{then} \quad Df_x(\mathcal{C}^c(x)) \subset \mathcal{C}^c(fx) \quad \text{and} \quad \forall v \in \mathcal{C}^c \ , \|\pi^c Df_x v\| \ge e^{\lambda} \|\pi^c v\|$$

and similar for $\mathcal{C}^s$, backward invariant, contract $\lambda_s < \min\{0, \lambda\}$

(A1)-(A3) satisfied by e.g.

$$u_t = \Delta u + g(u), \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0$$

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \subset\subset \mathbb{R}^n, \ u|_{\partial\Omega} = 0, \ \gamma > 0 \ .$$

**THEOREM .** Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{ graph } (h^c) \ , \ h^c : E^c \to E^s, \ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X, \ W^s(x) = \text{ graph}(h^s_x) \ , \ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

then $\text{Leb}(\theta(A)) \leq c \ \text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**

(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$

(c) : notion of ***"almost everywhere"*** in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$

**THEOREM .** Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{ graph } (h^c) \ , \ h^c : E^c \to E^s, \ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X, \ W^s(x) = \text{ graph}(h^s_x) \ , \ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

then $\text{Leb}(\theta(A)) \leq c \ \text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**




(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$

(c) : notion of ***"almost everywhere"*** in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$

**THEOREM** . Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{ graph } (h^c) \ , \ h^c : E^c \to E^s , \ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X, \ W^s(x) = \text{ graph}(h^s_x) \ , \ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

then $\text{Leb}(\theta(A)) \le c \, \text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**

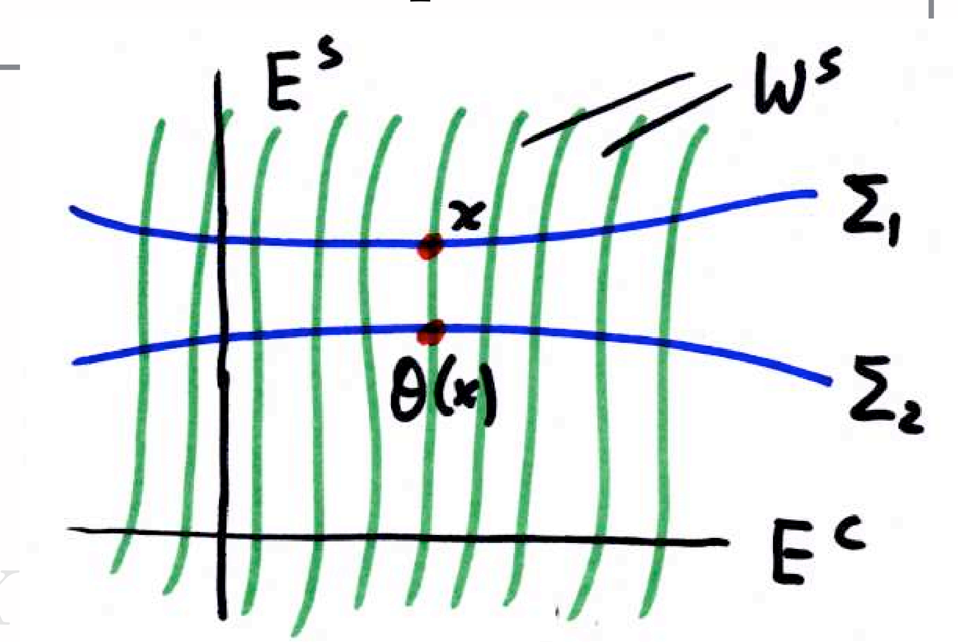
(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$

(c) : notion of ***"almost everywhere"*** in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$



**THEOREM** . Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{graph}\ (h^c)\ ,\ h^c : E^c \to E^s,\ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X,\ W^s(x) = \text{graph}(h^s_x)\ ,\ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

then $\text{Leb}(\theta(A)) \le c\ \text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**

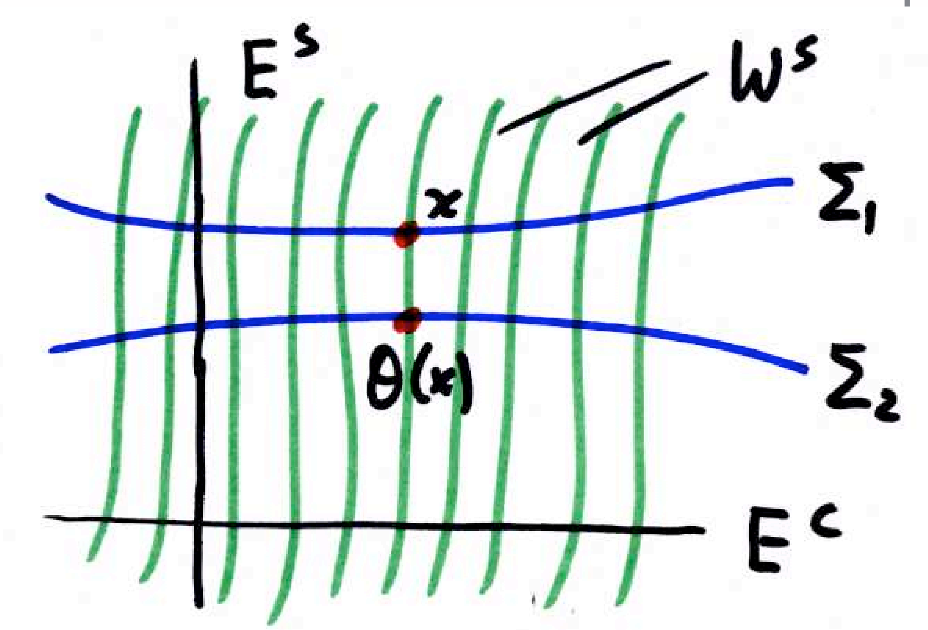
(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$

(c) : notion of ***"almost everywhere"*** in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$



**THEOREM .** Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{graph}\ (h^c)\ ,\ h^c : E^c \to E^s,\ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X,\ W^s(x) = \text{graph}(h^s_x)\ ,\ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

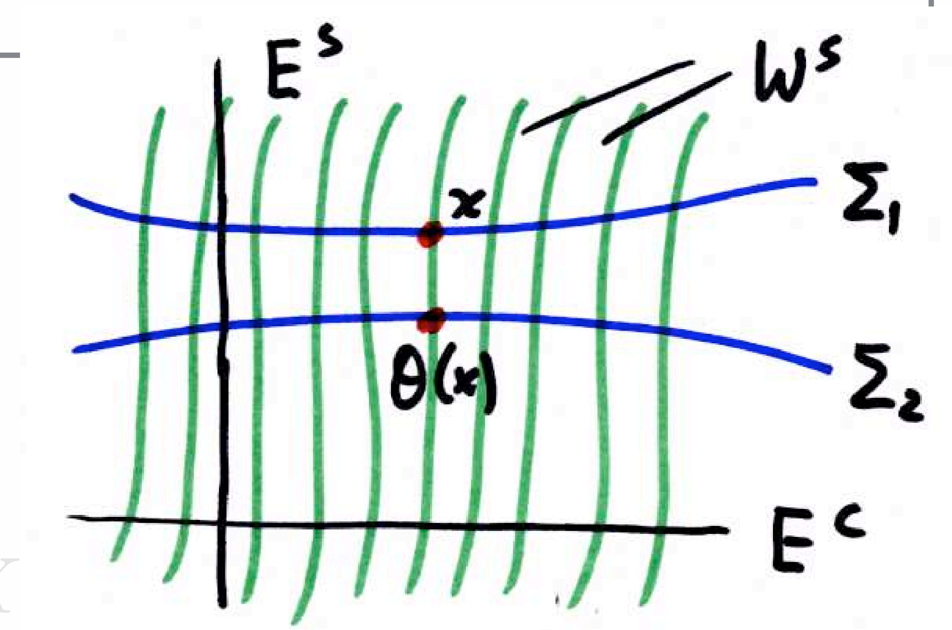
then $\text{Leb}(\theta(A)) \le c\, \text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**

(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$




(c) : notion of ***"almost everywhere"*** in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$

**THEOREM** . Under conditions (A1)-(A3),

(a) Existence of center manifold $W^c$ [known]

$$W^c = \text{graph}\,(h^c)\ ,\ h^c : E^c \to E^s,\ C^{1+\alpha}$$

(b) Existence of stable $W^s$ foliation [known]

For each $x \in X,\ W^s(x) = \text{graph}(h^s_x)\ ,\ h^s_x : E^s \to E^c$

s.t. $f(W^s(x)) \subset W^s(fx)$ , contract by $\lambda_s$ along $W^s$ leaves

(c) Absolute continuity of $W^s$-foliation in the case $\dim(W^c) < \infty$

i.e. if $\Sigma_1, \Sigma_2 =$ disks transversal to $W^s$, (Lian-Young-Zeng 2013)

and $\theta : \Sigma_1 \to \Sigma_2$ is holonomy along $W^s$-leaves,

then $\text{Leb}(\theta(A)) \le c\,\text{Leb}(A)$ for all Borel $A \subset \Sigma_1$ .

**Interpretation :**

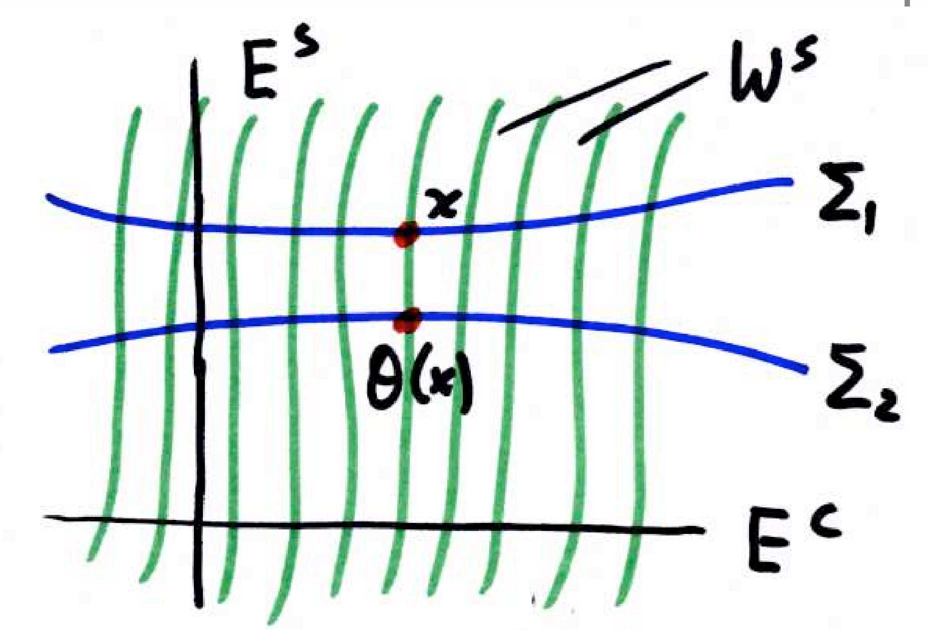
(a) : large-time dynamics near finite dim mfd

(b) : for each $u_0 \in X$ , $\exists v_0 \in W^c$ s.t.

$\|u(t) - v(t)\| \to 0$ exp fast as $t \to \infty$

(c) : notion of "***almost everywhere***" in $X$

Leb measure class on $W^c$ passed to k-dim'l mfds transversal to $W^s$



## Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of ``**a.e.**'' makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : shear induced chaos** **(Young et al 2000s)**

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

## Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : shear induced chaos** (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

## Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : shear induced chaos** (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : shear induced chaos** (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : shear induced chaos** (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

## Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

**Idea : *shear induced chaos*** (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space

Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

## Setting 2. SRB measures

**THEOREM** (Blumenthal-Young 2015) Consider general $(f, \mu)$ as in Part II. Assume $\mu$ is SRB with no 0 Lyap exp. Then $W^s$-foliation is abs cts.

**Interpretation** : notion of **``a.e.''** makes sense in neighborhood of attractor.

**Remarks** : In finite dim,

(1) SRB measures are *believed* to be present for many chaotic attractors, but *proving* is challenging (except where exp & contr directions are separated)

(2) Progress made for *rank-one attractors* $(\dim E^u = 1)$, which occur often following a system's loss of stability. (Wang-Young, 2002-08)

## Sample result #2. Example of an attractor with observable chaos

phenomenon occurs in finite as well as infinite dim (ODE or PDE)

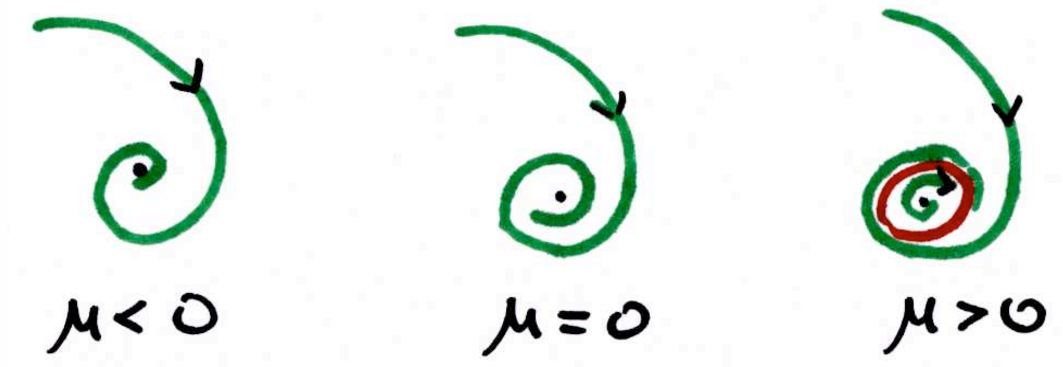
### Idea : *shear induced chaos* (Young et al 2000s)

**Unforced system** : simple dynamics, some ``shearing'' in phase space
Here : Hopf bifurcation, limit cycle following loss of stability

**Periodic forcing** : magnifies shear to stretch and fold phase space, producing ``strange attractor'' with open set of pos Lyap exp "a.e." in open set

## Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




 = limit cycle,
radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + k_2(\mu) z^3 \bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\text{Im } k_1(0)}{-\text{Re } k_1(0)}$

generic : $\text{Re } k_1(0), \ \text{Im } k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \ , \quad u \in X$$

Unforced system : assume $-A_\mu$ sectorial, generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

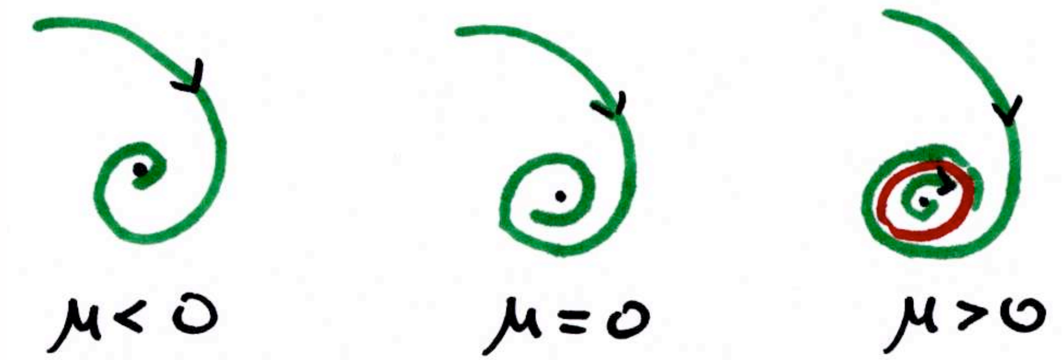



THEN for $\boxed{|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp ``a.e.''**

## Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$





= limit cycle,
radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + k_2(\mu) z^3 \bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\ k_1(0)}{-\mathrm{Re}\ k_1(0)}$

generic :
$\mathrm{Re}\ k_1(0),\ \mathrm{Im}\ k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \ , \quad u \in X$$

Unforced system : assume $-A_\mu$ sectorial, generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

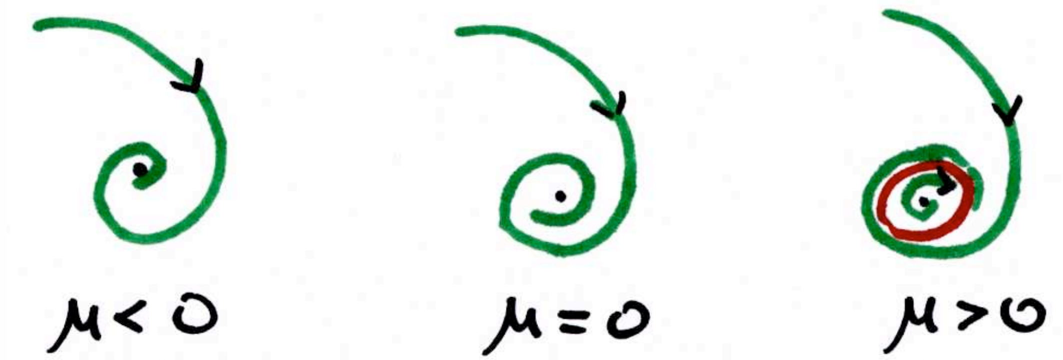



THEN for $\boxed{|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp ``a.e.''**

## Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




 = limit cycle, radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + k_2(\mu) z^3 \bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\ k_1(0)}{-\mathrm{Re}\ k_1(0)}$

generic : $\mathrm{Re}\ k_1(0),\ \mathrm{Im}\ k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \ , \quad u \in X$$

Unforced system : assume $-A_\mu$ sectorial, generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

$P_T(t)$

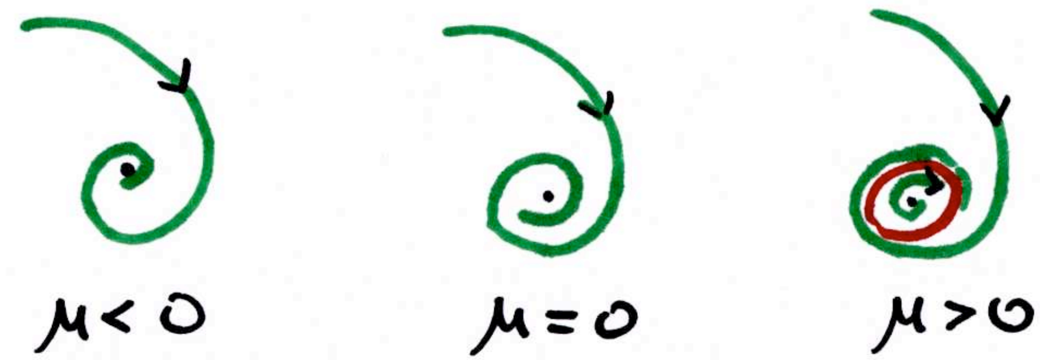



THEN for $|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp ``a.e.''**

## Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




 = limit cycle, radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + k_2(\mu) z^3 \bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\ k_1(0)}{-\mathrm{Re}\ k_1(0)}$

generic : $\mathrm{Re}\ k_1(0),\ \mathrm{Im}\ k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \ , \qquad u \in X$$

Unforced system : assume generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

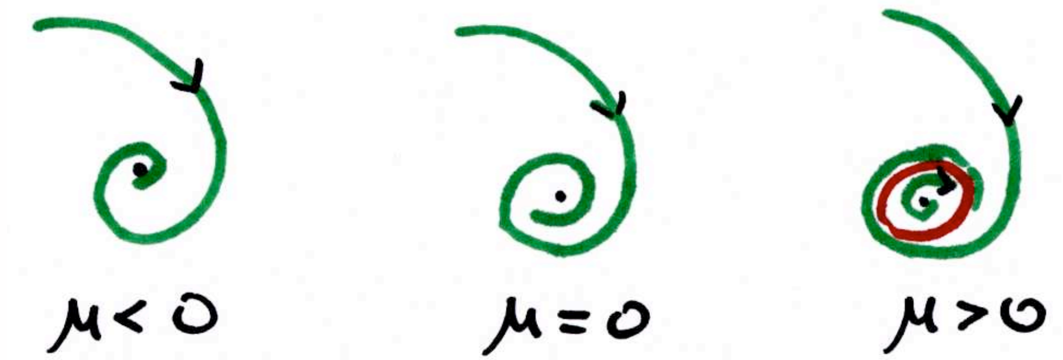



THEN for $\boxed{|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp "a.e."**

## Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




 = limit cycle,
radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu)z + k_1(\mu)z^2\bar{z} + k_2(\mu)z^3\bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\ k_1(0)}{-\mathrm{Re}\ k_1(0)}$

generic :
$\mathrm{Re}\ k_1(0),\ \mathrm{Im}\ k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

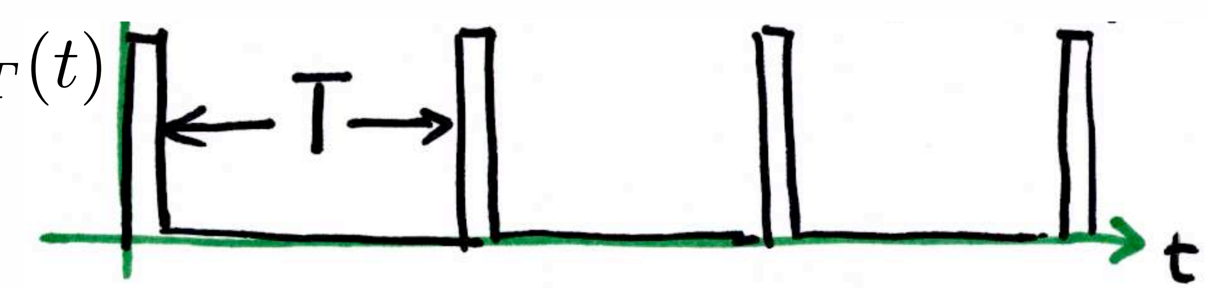
$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u)P_T(t) \ , \quad u \in X$$

Unforced system : assume generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth
$\pi^c\kappa(0) \neq 0$

$P_T(t)$

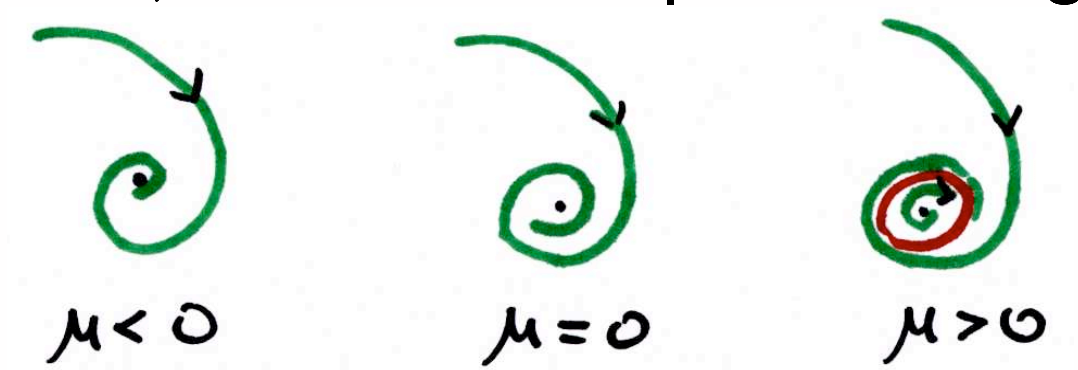



THEN for $\boxed{|\tau| \cdot \|\pi^c\kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp ``a.e.''**

Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \ \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




= limit cycle, radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu)z + k_1(\mu)z^2\bar{z} + k_2(\mu)z^3\bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\ k_1(0)}{-\mathrm{Re}\ k_1(0)}$

generic : $\mathrm{Re}\ k_1(0),\ \mathrm{Im}\ k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

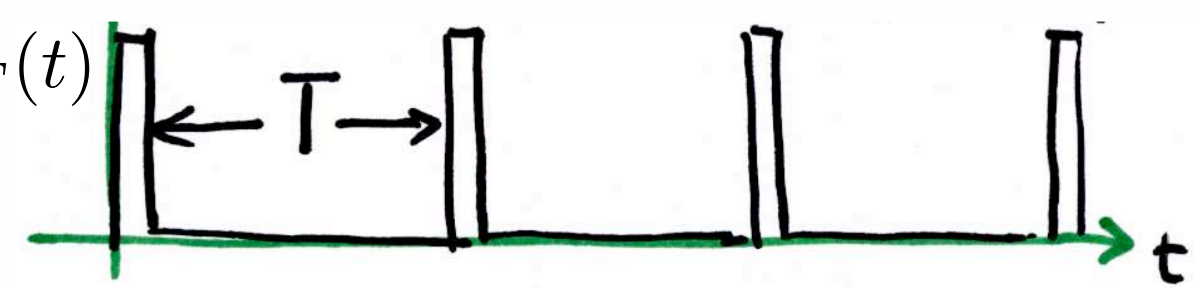
$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \ , \quad u \in X$$

Unforced system : assume generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

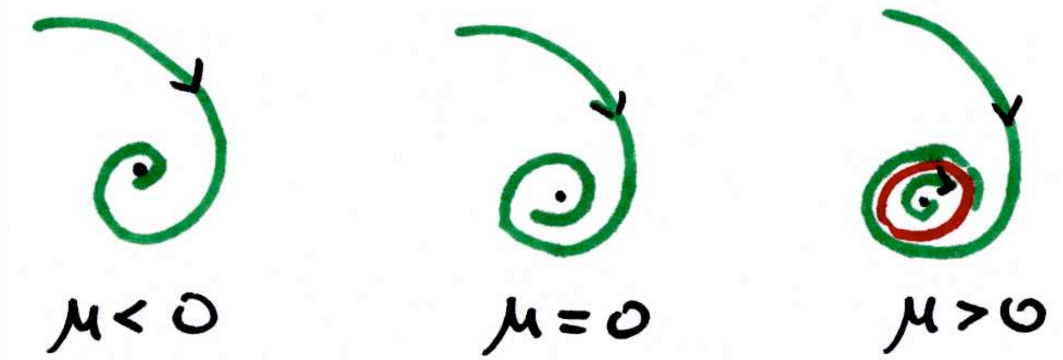



THEN for $\boxed{|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

Consequently, there is an open set in $X$ with pos Lyap exp "a.e."

Generic supercritical Hopf bifurcations i.e.

$\dot{x} = h_\mu(x)$ , $h_\mu(0) = 0 \; \forall \mu$ , pair of cx eigenvalues cross Im-axis at $\mu = 0$




 = limit cycle,
radius $\sim \sqrt{\mu}$

Normal form : $\dot{z} = k_0(\mu) z + k_1(\mu) z^2 \bar{z} + k_2(\mu) z^3 \bar{z}^2 + \cdots$

Introduce twist number $\tau = \dfrac{\mathrm{Im}\; k_1(0)}{-\mathrm{Re}\; k_1(0)}$

generic :
$\mathrm{Re}\; k_1(0), \; \mathrm{Im}\; k_1(0) \neq 0$

**THEOREM** (Lu-Wang-Young 2013) $X =$ Hilbert space

$$\partial_t u = A_\mu + f_\mu(u) \quad + \quad \kappa(u) P_T(t) \; , \quad u \in X$$

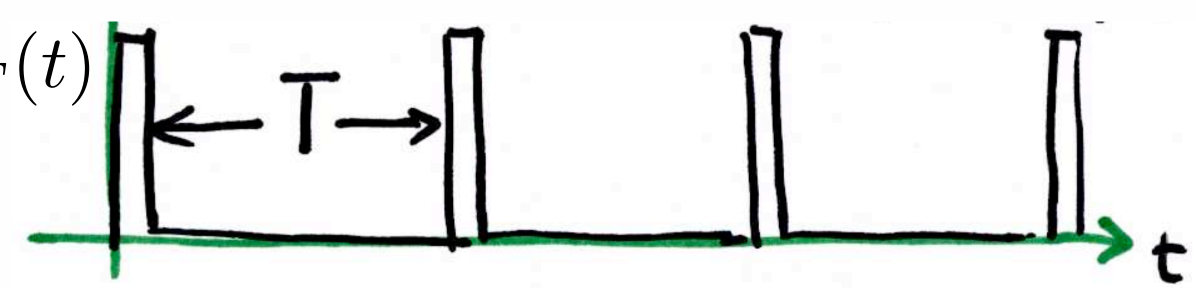
Unforced system : assume generic supercrit Hopf bif at $\mu = 0$

Forcing (last term) :

$\kappa : X \to X$ arbitrary, smooth

$\pi^c \kappa(0) \neq 0$

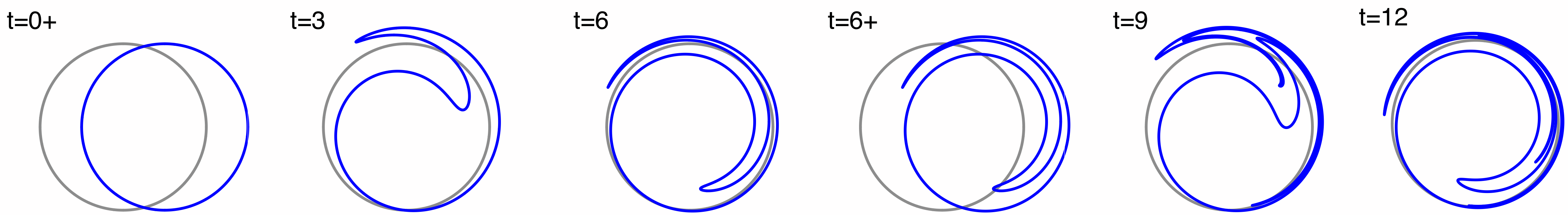
$P_T(t)$




THEN for $\boxed{|\tau| \cdot \; \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}}$ large enough, there is a pos Leb meas set of $T \ll 1$ for which the flow-map $\Phi_{\mu,T}$ has an attractor w/ SRB measure.

**Consequently, there is an open set in $X$ with pos Lyap exp "a.e."**

## Geometric meaning of $|\tau| \cdot \; \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}$




e.g. periodically forced *Brusselator* (autocatalytic chemical reaction) near Hopf bifurcation

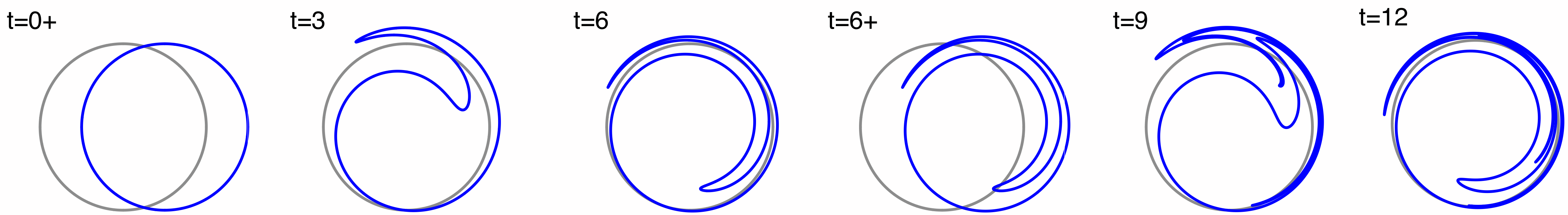
$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \\ \qquad\qquad\qquad\qquad \text{+ periodic forcing} \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases}$$




For suitable $\mu, T$ ,

- $\Phi_{\mu,T}$ has pos Lyap exp Leb-a.e. on 2D center manifold $W^c_{\mu,T}$ ,
- carried by strong codim-2 $W^{ss}$ foliation to ``a.e.'' in an open set in inf dim phase space

# Geometric meaning of $|\tau| \cdot \; \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}$




e.g. periodically forced *Brusselator* (autocatalytic chemical reaction)
near Hopf bifurcation

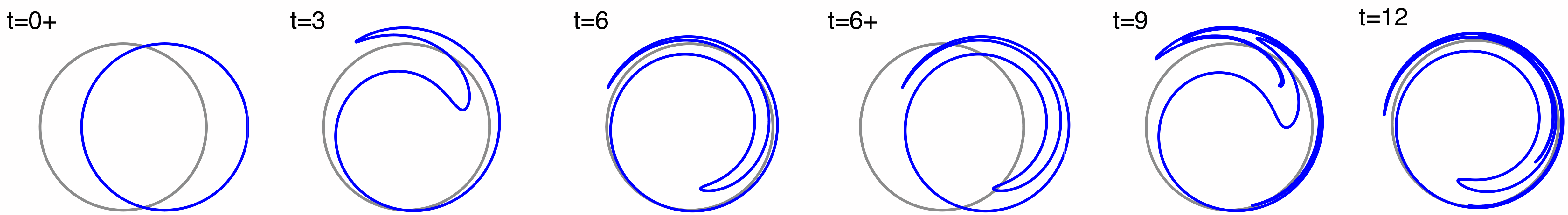
$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \\ \qquad\qquad\qquad\qquad\qquad + \text{ periodic forcing} \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases}$$




For suitable $\mu, T$,

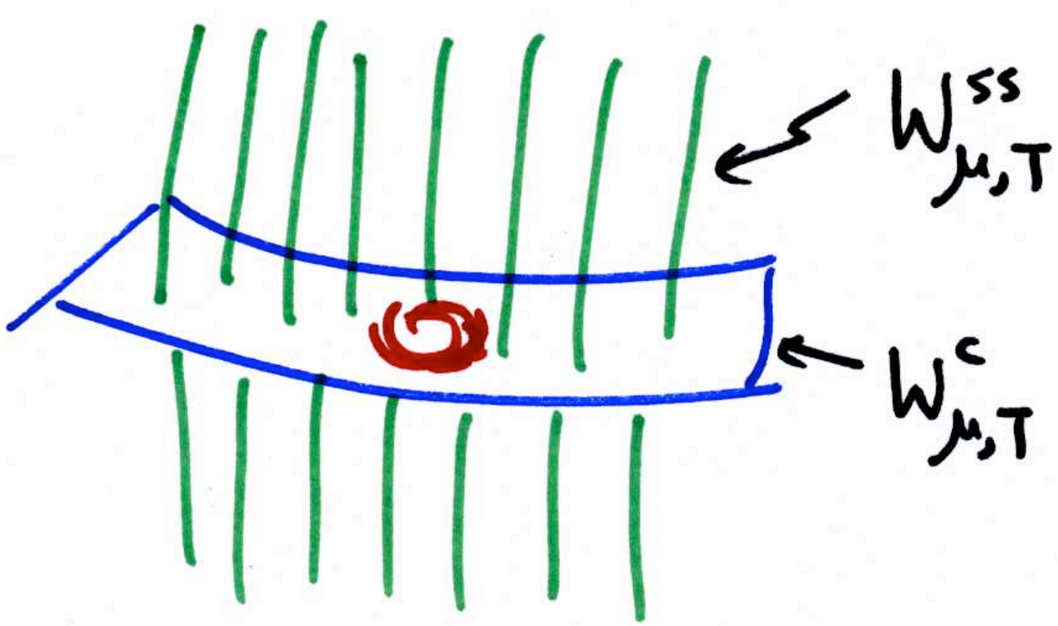
- $\Phi_{\mu,T}$ has pos Lyap exp Leb-a.e. on 2D center manifold $W^c_{\mu,T}$,
- carried by strong codim-2 $W^{ss}$ foliation to ``a.e.'' in an open set in inf dim phase space

## Geometric meaning of $|\tau| \cdot \ \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}$




e.g. periodically forced *Brusselator* (autocatalytic chemical reaction) near Hopf bifurcation

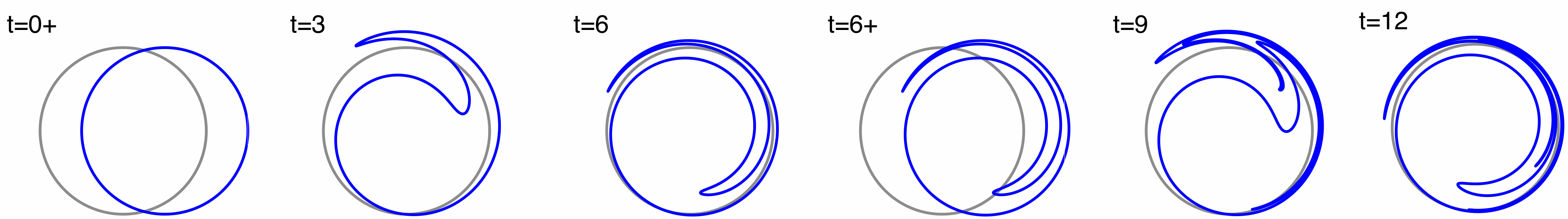
$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{+ periodic forcing} \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases}$$




For suitable $\mu, T$,

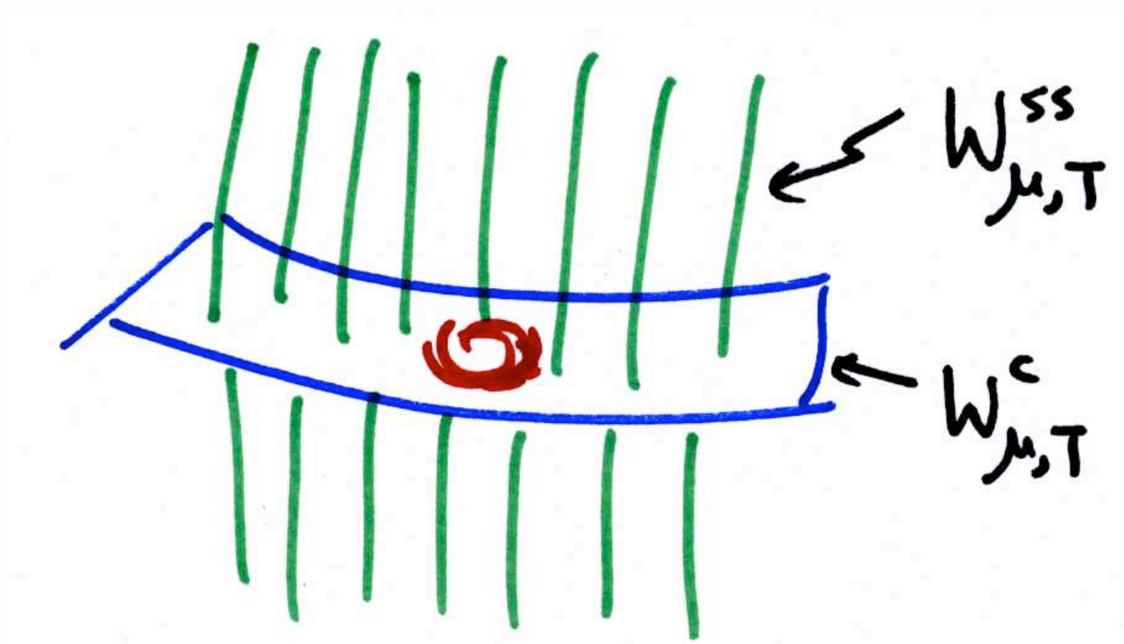
- $\Phi_{\mu,T}$ has pos Lyap exp Leb-a.e. on 2D center manifold $W^c_{\mu,T}$ ,
- carried by strong codim-2 $W^{ss}$ foliation to ``a.e.'' in an open set in inf dim phase space

Geometric meaning of $|\tau| \cdot \; \|\pi^c \kappa(0)\| \cdot \mu^{-\frac{1}{2}}$




e.g. periodically forced *Brusselator* (autocatalytic chemical reaction)
near Hopf bifurcation

$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \quad \text{+ periodic forcing} \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases}$$




For suitable $\mu, T$ ,

- $\Phi_{\mu,T}$ has pos Lyap exp Leb-a.e. on 2D center manifold $W^c_{\mu,T}$ ,
- carried by strong codim-2 $W^{ss}$ foliation to ``a.e.'' in an open set in inf dim phase space

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)
- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .
- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

- In finite dim : positive Leb meas sets = observable events
- In infinite dim ??? Introduced notion of ``almost everywhere''

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)

- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .

- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

  Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

  - In finite dim : positive Leb meas sets = observable events
  - In infinite dim ??? Introduced notion of ``almost everywhere''

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)
- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .
- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

- In finite dim : positive Leb meas sets = observable events
- In infinite dim ??? Introduced notion of ``almost everywhere"

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)
- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .
- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

- In finite dim : positive Leb meas sets = observable events
- In infinite dim ??? Introduced notion of ``almost everywhere''

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)

- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .

- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

- In finite dim : positive Leb meas sets = observable events
- In infinite dim ??? Introduced notion of ``almost everywhere''

# Summary

- In finite dim, there is a fairly well developed smooth ergodic theory (of diffeomorphisms and of flows generated by ODEs)
- I have tried to report on extensions of this theory to infinite dim, to settings that include dissipative PDEs .
- For inf dim systems with a finite dim character, e.g. finite dim $E^u$ , technical issues largely resolvable, and theory carried over thus far.

  Sample results : entropy , Lyap exp , horseshoes , SRB measures, absolute continuity of invariant foliations , strange attractors

- Inherent issue in *deterministic* dynamics, both finite/infinite dim : *which invariant measures reflect what we see/are more important ?*

  - In finite dim : positive Leb meas sets = observable events
  - In infinite dim ??? Introduced notion of ``almost everywhere”