Data Assimilation: New Challenges in Random and Stochastic Dynamical Systems

Daniel Sanz-Alonso & Andrew Stuart

D Blömker (Augsburg), D Kelly (NYU), KJH Law (KAUST),
A. Shukla (Warwick), KC Zygalakis (Southampton)

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Enabling Quantification of EQUIP Uncertainty for Inverse Problems

THE UNIVERSITY OF WARWICK

Outline

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Signal

Consider the following map on Hilbert space $\big(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|\big)$:

Signal Dynamics

$$v_{j+1} = \Psi(v_j), \qquad v_0 \sim \mu_0.$$

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Assume **dissipativity**:

Absorbing Set

Compact B in $\mathcal{H}$ with the property that, for $|v_0| \leq R$, there is $J = J(R) > 0$ such that, for all $j \geq J$, $v_j \in B$.

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Limited predictability:

Global Attractor

$$d(v_j, \mathcal{A}) \to 0, \text{ as } j \to \infty.$$

Signal and Observation

Random initial condition:

Signal Process

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Observations, **partial and noisy**, $P : \mathcal{H} \to \mathbb{R}^J$:

Observation Process

$$y_{j+1} = Pv_{j+1} + \epsilon\xi_{j+1}, \qquad \mathbb{E}\xi_j = 0, \quad \mathbb{E}|\xi_j|^2 = 1, \text{ i.i.d. w/pdf } \rho.$$

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Filter: probability distribution of v_j given observations to time j:

Filter

$$\mu_j(A) = \mathbb{P}\big(v_j \in A | \mathcal{F}_j\big), \quad \mathcal{F}_j = \sigma(y_1, \ldots, y_j).$$

Signal and Observation: Control Unpredictability?

Pushforward under dynamics:

Signal Process

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Incorporate **observations via Bayes' Theorem:**

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$$\mu_{j+1}(A) = \frac{\int_A \rho\big(\epsilon^{-1}(y_{j+1} - Pv)\big)\widehat{\mu}_{j+1}(dv)}{\int_{\mathcal{H}} \rho\big(\epsilon^{-1}(y_{j+1} - Pv)\big)\widehat{\mu}_{j+1}(dv)}.$$

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When is the **filter predictable**:

Filter Accuracy

$$\mu_j \approx \delta_{v_j^\dagger} \text{ as } j \to \infty.$$

Goal (Cerou [5], SIAM J. Cont. Opt. 2000)

Key Question: For which Ψ and P does the filter μ^j concentrate on the true signal, up to error ϵ, in the large-time limit?

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Key Question: For which Ψ and P does the filter μ^j concentrate on the true signal, up to error ϵ, in the large-time limit?

Key Problem: Ψ may expand

View P as a **projection** on $\mathcal{H}$. Define $Q = I - P$.

Key Idea: $Q\Psi$ should contract

A Large Class of Examples

Geophysical Applications

$$\frac{dv}{dt} + Au + B(u,u) = f.$$

Dissipative with energy conserving nonlinearity

- $\exists \lambda > 0 : \langle Av, v \rangle \geq \lambda |v|^2.$
- $\langle B(v,v), v \rangle = 0.$
- $f \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{H}).$

Examples

Lorenz '63
Lorenz '96
Incompressible 2D Navier-Stokes equation on a torus

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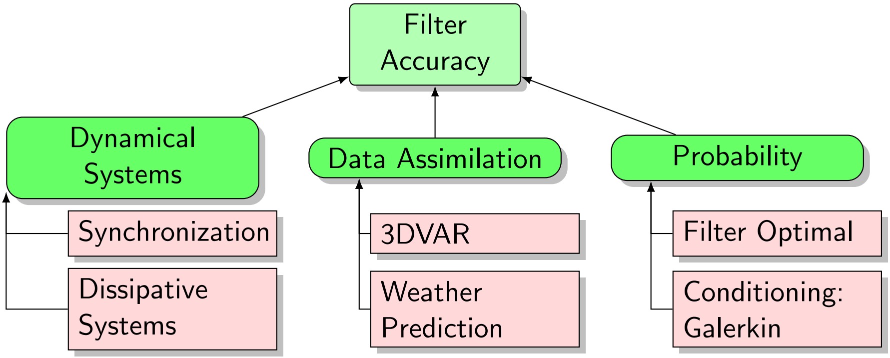
Filter Accuracy
Dynamical Systems
Data Assimilation
Probability
Synchronization
Dissipative Systems
3DVAR
Weather Prediction
Filter Optimal
Conditioning: Galerkin

Idea 1: Synchronization (Foias and Prodi [7], RSM Padova 1967 Pecora and Carroll [13], PRL 1990.)

Truth $v^\dagger = (p^\dagger, q^\dagger)$	Synchronization Filter $m = (p, q)$
$p^\dagger_{j+1} = P\Psi(p^\dagger_j, q^\dagger_j),$	$p_{j+1} = p^\dagger_{j+1},$
$q^\dagger_{j+1} = Q\Psi(p^\dagger_j, q^\dagger_j),$	$q_{j+1} = Q\Psi(p^\dagger_j, q_j);$
$- - -$	$- - -$
$v^\dagger_{j+1} = \Psi(v^\dagger_j),$	$m_{j+1} = Q\Psi(m_j) + p^\dagger_{j+1}.$

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$$m_{j+1} = Q\Psi(m_j) + p^\dagger_{j+1}.$$

Synchronization for various chaotic dynamical systems (including the three canonical examples above [8, 13, 4, 14]):

$$|m_j - v^\dagger_j| \to 0, \text{ as } j \to \infty.$$

Idea 2: 3DVAR (Lorenc [12] Q. J. R. Met. Soc 1986)

Cycled 3DVAR Filter. $|\cdot|_A = |A^{-\frac{1}{2}} \cdot|$.

$$m_{j+1} = \operatorname{argmin}_{m \in \mathcal{H}} \{|m - \Psi(m_j)|_C^2 + \epsilon^{-2}|y_{j+1} - Pm|_\Gamma^2\}.$$

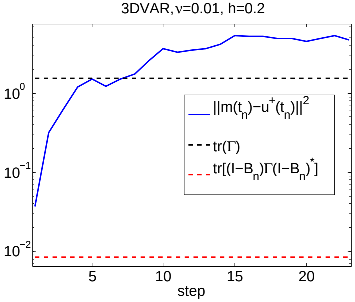
Solve Variational Equations (with $C = \epsilon^2(\eta^{-2}\Gamma P + Q)$)

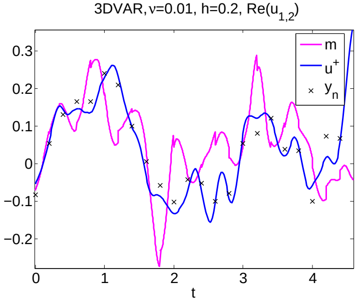
$$m_{j+1} = (I-K)\Psi(m_j) + Ky_{j+1}, \quad K = (1+\eta^2)^{-1}P,$$

Variance Inflation (from weather prediction) $\eta \ll 1$

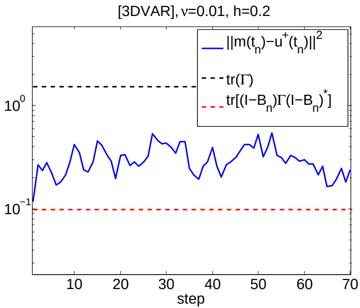
$$m_{j+1} = Q\Psi(m_j) + Py_{j+1}, \quad \eta = 0.$$ **Synchronization Filter.**

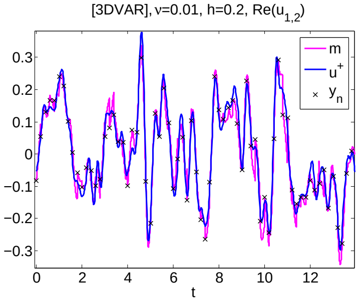
Inaccurate: η too large. (NSE torus) Law and S [10], Monthly Weather Review, 2012





Accurate: smaller η. (NSE torus) Law and S [10], Monthly Weather Review, 2012





Idea 3: Filter Optimality (Folklore, but see e.g. Williams · · ·)

Recall $\mathcal{F}_j = \sigma(y_1, \dots, y_j)$ and define the mean of the filter:

$$\hat{v}_j := \mathbb{E}(v_j|\mathcal{F}_j) = \mathbb{E}^{\mu_j}(v_j).$$

Use Galerkin orthogonality wrt conditional expectation

For any $\mathcal{F}_j$ measurable m_j :

$$\mathbb{E}|v_j - \hat{v}_j|^2 \leq \mathbb{E}|v_j - m_j|^2.$$

Take m_j from **3DVAR** to get bounds on the mean of the filter.
Similar bounds apply to the variance of the filter. (Not shown.)

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Assumptions

There are two equivalent Hilbert spaces:
$\big(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|\big)$ and $\big(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}}, \|\cdot\|\big)$:

Assumption 1: Absorbing Ball Property

There is $R_0 > 0$ such that:

- for $B(R_0) := \{x \in \mathcal{H} : |x| \le R_0\}$, $\Psi(B(R_0)) \subset B(R_0)$;
- for any bounded set $S \subset \mathcal{H}$ $\exists J = J(S) : \Psi^J(S) \subset B(R_0)$.

Assumption 2: Squeezing Property

There is $\alpha(R_0) \in (0,1)$ such that, for all $u, v \in B(R_0)$,

$$\|Q(\Psi(u) - \Psi(v))\|^2 \le \alpha(\mathbf{R_0})\|u - v\|^2.$$

Theorem (Sanz-Alonso and S, 2014, [15])

Let Assumptions 1,2 hold. Then there is a constant $c > 0$ independent of the noise strength ϵ such that

$$\limsup_{j\to\infty} \mathbb{E}|v_j - \hat{v}_j|^2 \le c\epsilon^2$$

.

Idea of proof:

- Fix $m_0 \in B(R_0)$ and let $\mathcal{P}$ denote the $\mathcal{H}-$projection onto $B(R_0)$. Define the **modified 3DVAR**:

$$m_{j+1} = \mathcal{P}\big(Q\Psi(m_j) + y_{j+1}\big).$$

- Prove

$$\limsup_{j\to\infty} \mathbb{E}|v_j - m_j|^2 \le c\epsilon^2.$$

- Use the L^2 optimality of the filtering distribution.

Idea of proof (sketch, Ψ globally Lipschitz):

$$
\begin{aligned}
m_{j+1} &= Q\Psi(m_j) && + \overbrace{P\Psi(v_j) + \epsilon\xi_{j+1}}^{\mathbf{y_{j+1}}},\\
v_{j+1} &= Q\Psi(v_j) && + P\Psi(v_j).
\end{aligned}
$$

Subtract and use independence plus contractivity of $Q\Psi$:

$$
\begin{aligned}
\mathbb{E}\|v_{j+1} - m_{j+1}\|^2 &= \mathbb{E}\|Q\left(\Psi(v_j) - \Psi(m_j)\right) - \epsilon\xi_{j+1}\|^2\\
&\le \mathbb{E}\|Q\left(\Psi(v_j) - \Psi(m_j)\right)\|^2 + \epsilon^2\mathbb{E}\|\xi_{j+1}\|^2\\
&\le \alpha\mathbb{E}\|v_j - m_j\|^2 + \epsilon^2\mathbb{E}\|\xi_{j+1}\|^2.
\end{aligned}
$$

Use Gronwall.

Lorenz '63 (uses noiseless synchronization filter analysis in Hayden, Olson and Titi [8], Physica D 2011.)

$$\begin{aligned}
\frac{dv^{(1)}}{dt} &+ a(v^{(1)} - v^{(2)}) &&= 0\\
\frac{dv^{(2)}}{dt} &+ av^{(1)} + v^{(2)} + v^{(1)}v^{(3)} &&= 0\\
\underbrace{\frac{dv^{(3)}}{dt}}_{\frac{\mathbf{dv}}{\mathbf{dt}}} &+ \underbrace{bv^{(3)}}_{\mathbf{Av}} \underbrace{- v^{(1)}v^{(2)}}_{\mathbf{B(v,v)}} &&= \underbrace{-b(r+a)}_{\mathbf{f}}
\end{aligned}$$

Observation matrix

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theory applicable with $\|\cdot\|^2 := |P\cdot|^2 + |\cdot|^2$ for h sufficiently small: [8], [11].

Lorenz '96 (Law, Sanz-Alonso, Shukla and S [14], arXiv 2014.)

Consider the following system, subject to the periodicity boundary conditions $v_0 = v_{3J}, v_{-1} = v_{3J-1}, v_{3J+1} = v_1$:

$$\underbrace{\frac{dv^{(j)}}{dt}}_{\frac{\mathbf{dv}}{\mathbf{dt}}} + \underbrace{v^{(j)}}_{\mathbf{Av}} + \underbrace{v^{(j-1)}(v^{(j+1)} - v^{(j-2)})}_{\mathbf{B(v,v)}} = \underbrace{F}_{\mathbf{f}}, \quad j = 1, 2, \ldots, 3J.$$

Observation matrix P: observe 2 out of every 3 points. Theory applicable with $\|\cdot\|^2 := |P\cdot|^2 + |\cdot|^2$ for h sufficiently small: [14].

2D Navier-Stokes Equation (again uses analysis in Hayden, Olson and Titi [8], Physica D 2011.)

P_{leray} denotes the Leray projector:

$$Au = -\nu P_{\mathrm{leray}} \Delta u, \qquad B(u,v) = \frac{1}{2} P_{\mathrm{leray}}[u \cdot \nabla v] + \frac{1}{2} P_{\mathrm{leray}}[v \cdot \nabla u].$$

Observation operator in (divergence-free) Fourier space:

$$Pu = \sum_{|k| \le k_{\mathrm{max}}} u_k \frac{k^{\perp}}{|k|} e^{ik \cdot x}.$$

Theory applicable with $\mathcal{H} = \mathcal{V} := H^1_{\mathrm{div}}(\mathbb{T}^2)$ and k_{max} sufficiently large/h sufficiently small: [3], [8].

Summary of Examples

Observations control unpredictability in these cases:

ODE	Dimension of v	Rank(P)
Lorenz '63	3	1
Lorenz '96	3J	2J
NSE on torus	∞	Finite

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Generalize 3DVAR: The EnKF. Evensen [6] Journal of Geophysical Research 1994.

Ensemble Kalman Filter. For $n = 1, \dots, N$:

$$v_{j+1}^{(n)} = \mathrm{argmin}_{v \in \mathcal{H}} \{ |v - \Psi(v_j^{(n)})|_{C_j}^2 + \epsilon^{-2} |y_{j+1}^{(n)} - Pv|_\Gamma^2 \}.$$

Empirical Covariance

$$\bar{v}_j = \frac{1}{N} \sum_{n=1}^{N} v_j^{(n)}, \quad C_j = \frac{1}{N} \sum_{n=1}^{N} \big(v_j^{(n)} - \bar{v}_j\big) \otimes \big(v_j^{(n)} - \bar{v}_j\big).$$

Perturbed Observations

$$y_{j+1}^{(n)} = y_{j+1} + \epsilon \xi_j^{(n)}, \quad \xi_j^{(n)} \text{ i.i.d. w/pdf } \rho.$$

S(P)DE Limits with Brett et al [3], Blömker et al 2012 [2], Kelly et al 2014 [9]

High Frequency Data Limit – 3DVAR

$$\frac{dm}{dt} + Am + B(m,m) + CP^*\Gamma^{-1}\Big(P(m-v) + \epsilon\Gamma^{\frac{1}{2}}\frac{dW}{dt}\Big) = f$$

S(P)DE Limits with Brett et al [3], Blömker et al 2012 [2], Kelly et al 2014 [9]

High Frequency Data Limit – 3DVAR

$$\frac{dm}{dt} + Am + B(m,m) + CP^*\Gamma^{-1}\Big(P(m-v) + \epsilon\Gamma^{\frac{1}{2}}\frac{dW}{dt}\Big) = f$$

High Frequency Data Limit – Ensemble Kalman Filter

$$\frac{dv^{(n)}}{dt} + Av^{(n)} + B(v^{(n)}, v^{(n)}) + CP^*\Gamma^{-1}\Big(P(v^{(n)} - v) + \epsilon\Gamma^{\frac{1}{2}}\frac{dW^{(n)}}{dt}\Big) = f,$$

$$\overline{v} = \frac{1}{J}\sum_{j=1}^{J} v^{(n)}, \quad C = \frac{1}{J}\sum_{j=1}^{J}(v^{(n)} - \overline{v}) \otimes (v^{(n)} - \overline{v}).$$

S(P)DE Accuracy see also Azouani, Olson and Titi 2014 [1] and Tong, Majda, Kelly 2015 [16] .

Theorem (3DVAR Accurate, with Blömker 2012 et al [2])

Under similar assumptions to the discete case there is a constant $c > 0$ independent of the noise strength ϵ such that

$$\limsup_{t\to\infty} \mathbb{E}|v - m|^2 \le c\epsilon^2$$

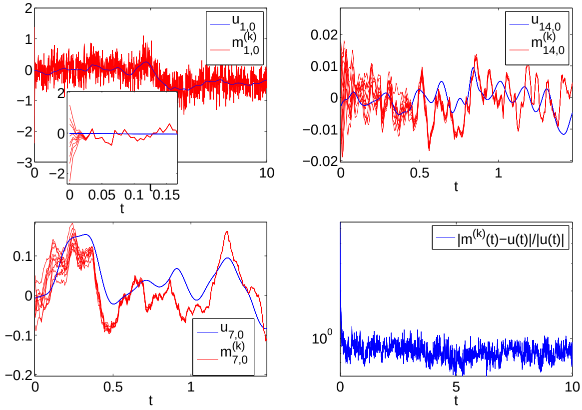
.

Theorem (EnKF Well-Posed, with Kelly et al [9])

Let Assumptions 1 hold and P=I. Then there is a constant $c > 0$ independent of the noise strength ϵ such that

$$\sup_{t\in[0,T]} \sum_{n=1}^{N} \mathbb{E}|v^{(n)}(t)|^2 \le C(T)\big(1 + \mathbb{E}|v^{(n)}(0)|^2\big).$$

SPDE Inaccurate (NSE Torus) (Blömker et al [2])



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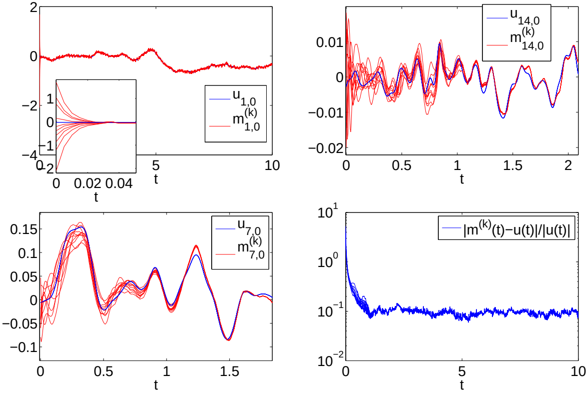

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- The emerging new field, in which **model and data are analyzed simultaneously**, will lead to interesting new mathematics over the next century.

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- The emerging new field, in which **model and data are analyzed simultaneously**, will lead to interesting new mathematics over the next century.
- **Data Assimilation** needs input from **Dynamical Systems**.

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MATLAB files and book chapters freely available:

http://tiny.cc/damat

http://www2.warwick.ac.uk/fac/sci/maths/people/staff/andrew_stuart/