Effective Models for Ginzburg-Landau Vortices

Sylvia Serfaty

Université P. et M. Curie Paris 6, Laboratoire Jacques-Louis Lions

Equadiff-2015, Lyon, July 6, 2015

Outline

The Ginzburg-Landau equations

$u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$

$$\boxed{-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Ginzburg-Landau equation (GL)}$$

$$\boxed{\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{parabolic GL equation (PGL)}$$

$$\boxed{i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Gross-Pitaevskii equation (GP)}$$

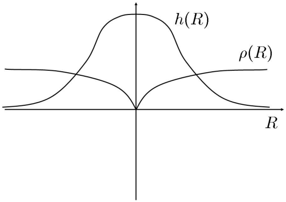
Associated energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

Models: superconductivity, superfluidity, Bose-Einstein condensates, nonlinear optics

Vortices

- in general $|u| \leq 1$, $|u| \simeq 1$ = superconducting/superfluid phase, $|u| \simeq 0$ = normal phase
- u has zeroes with nonzero degrees = **vortices**
- $u = \rho e^{i\varphi}$, characteristic length scale of $\{\rho < 1\}$ is ε = vortex core size



- degree of the vortex at x_0:

$$\frac{1}{2\pi} \int_{\partial B(x_0,r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

- In the limit $\varepsilon \to 0$ vortices become *points*, (or curves in dimension 3).

Solutions of (GL), bounded number N of vortices

- minimal energy

$$\min E_\varepsilon = \pi N|\log\varepsilon| + \min W + o(1) \quad \text{as} \quad \varepsilon \to 0$$

- u_ε minimizing E_ε has vortices all of degree $+1$ (or all -1) which converge to a minimizer of

$$W((x_1, d_1), \dots, (x_N, d_N)) = -\pi \sum_{i \neq j} d_i d_j \log|x_i - x_j| + \text{boundary terms...}$$

  "renormalized energy", Kirchhoff-Onsager energy (in the whole plane) [Bethuel-Brezis-Hélein '94]
- Some boundary condition needed to obtain nontrivial minimizers
- nonminimizing solutions: u_ε has vortices which converge to a critical point of W:

$$\nabla_i W(\{x_i\}) = 0 \quad \forall i = 1, \cdots N$$

  [Bethuel-Brezis-Hélein '94]

Solutions of (GL), bounded number N of vortices

- minimal energy

$$\min E_\varepsilon = \pi N|\log\varepsilon| + \min W + o(1) \quad \text{as} \quad \varepsilon \to 0$$

- u_ε minimizing E_ε has vortices all of degree $+1$ (or all -1) which converge to a minimizer of

$$W((x_1,d_1),\dots,(x_N,d_N)) = -\pi\sum_{i\neq j} d_i d_j \log|x_i - x_j| + \text{boundary terms...}$$

  "renormalized energy", Kirchhoff-Onsager energy (in the whole plane) [Bethuel-Brezis-Hélein '94]

- Some boundary condition needed to obtain nontrivial minimizers
- nonminimizing solutions: u_ε has vortices which converge to a critical point of W:

$$\nabla_i W(\{x_i\}) = 0 \quad \forall i = 1,\cdots N$$

  [Bethuel-Brezis-Hélein '94]

Dynamics, bounded number N of vortices

- For well-prepared initial data, $d_i = \pm 1$, solutions to (PGL) have vortices which converge (after some time-rescaling) to solutions to

$$\frac{dx_i}{dt} = -\nabla_i W(x_1, \dots, x_N)$$

[Lin '96, Jerrard-Soner '98, Lin-Xin '99, Spirn '02, Sandier-S '04]

- For well-prepared initial data, $d_i = \pm 1$, solutions to (GP)

$$\frac{dx_i}{dt} = -\nabla_i^\perp W(x_1, \dots, x_N) \qquad \nabla^\perp = (-\partial_2, \partial_1)$$

[Colliander-Jerrard '98, Spirn '03, Bethuel-Jerrard-Smets '08]

- All these hold up to collision time
- For (PGL), extensions beyond collision time and for ill-prepared data [Bethuel-Orlandi-Smets '05-07, S. '07]

A word about dimension 3 (or higher)

- Leading order of the energy becomes $\pi|d|L|\log\varepsilon|$ where L= length (or area) of vortex line (integer multiplicity rectifiable current)
- Minimizers/solutions to (GL) converge to length minimizing / stationary currents (= straight lines)
  [Rivière '95, Lin-Rivière '01, Sandier '01, Bethuel-Brezis-Orlandi '01, Jerrard-Soner '02]
- (PGL) $\rightarrow$ mean curvature motion (Brakke)
  [Bethuel-Orlandi-Smets '06]
- (GP) $\rightarrow$ binormal flow (partial results)
  [Jerrard '02]

The Ginzburg-Landau model with gauge

$$G_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla u - iAu|^2 + |\mathrm{curl}\, A - h_{\mathrm{ex}}|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

- $A : \Omega \to \mathbb{R}^2$ = gauge field
- $h = \mathrm{curl}\, A := \partial_2 A_1 - \partial_1 A_2$ induced magnetic field
- h_{ex}= intensity of applied magnetic field, causes vortices to appear, instead of an (artificial) boundary condition
- GL system

$$\begin{cases} -(\nabla - iA)^2 u = \frac{u}{\varepsilon^2}(1 - |u|^2) & \text{in } \Omega \\ -\nabla^\perp h = \langle iu, \nabla u - iAu \rangle & \text{in } \Omega \\ h = h_{\mathrm{ex}} & \text{on } \partial\Omega \\ (\nabla u - iAu) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

- + dynamical versions

Behavior of minimizers in terms of $h_{\rm ex}$

- there exists a first critical field $H_{c_1} \sim C_\Omega |\log \varepsilon|$ as $\varepsilon \to 0$
- for $h_{\rm ex} \le H_{c_1}$ no vortices
- for $h_{\rm ex} = H_{c_1}$ one vortex appears, degree $+1$, near a point p in the "center" of the domain
- for $h_{\rm ex} \ge H_{c_1} + c \log |\log \varepsilon|$ a second vortex appears
- vortices get added one by one for each increment of $\log |\log \varepsilon|$, they tend to minimize an effective interaction energy
- as soon as $h_{\rm ex} - H_{c_1} \gg \log |\log \varepsilon|$ the number N_ε of vortices is unbounded

[S. '98, Sandier-S '00-07]

Vorticity

- In the case $N_\varepsilon \to \infty$, describe the vortices via the **vorticity** :

  supercurrent

  $$j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \qquad \langle a, b \rangle := \frac{1}{2}(a\bar{b} + \bar{a}b)$$

  vorticity

  $$\mu_\varepsilon := \mathrm{curl}\, j_\varepsilon$$

- $\simeq$ vorticity in fluids, but quantized: $\mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i^\varepsilon}$
- $\exists$ a gauged-version
- $\frac{\mu_\varepsilon}{2\pi N_\varepsilon} \to \mu$ signed measure, or probability measure,

Mean-field model for energy minimizers

- Minimizers of G_ε have vorticity μ_ε, such that $\mu_\varepsilon/h_{\mathrm{ex}}$ converges to the minimizer μ_* of

$$\Phi_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2$$

where

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega \end{cases}$$

$$\lambda := \lim_{\varepsilon\to 0} \frac{h_{\mathrm{ex}}}{|\log\varepsilon|}.$$

- Minimal energy

$$\min G_\varepsilon \sim h_{\mathrm{ex}}^2 \Phi_\lambda(\mu_*).$$

- One can observe that

$$\mu_* \neq 0 \Leftrightarrow \lambda > \lambda_\Omega \qquad \rightsquigarrow \boxed{H_{c_1} \sim \lambda_\Omega |\log\varepsilon|}$$

[Sandier-S '00]

Mean-field model for energy minimizers

- Minimizers of G_ε have vorticity μ_ε, such that $\mu_\varepsilon/h_{\rm ex}$ converges to the minimizer μ_* of

$$\Phi_\lambda(\mu) = \frac{1}{2\lambda}\int_\Omega |\mu| + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2$$

where

$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega \\ h_\mu = 1 & \text{on } \partial\Omega \end{cases}$$

$$\lambda := \lim_{\varepsilon \to 0} \frac{h_{\rm ex}}{|\log \varepsilon|}.$$

- Minimal energy

$$\min G_\varepsilon \sim h_{\rm ex}^2 \Phi_\lambda(\mu_*).$$

- One can observe that

$$\mu_* \neq 0 \Leftrightarrow \lambda > \lambda_\Omega \qquad \rightsquigarrow \boxed{H_{c_1} \sim \lambda_\Omega |\log \varepsilon|}$$

[Sandier-S '00]

- μ_* corresponds to the solution of a free-boundary problem (obstacle problem), also like equilibrium measure in potential theory
- The optimal $\mu_* = -\Delta h_* + h_*$ is a uniform density on a subdomain $\omega_\lambda \subset \Omega$, which grows as $\lambda > \lambda_\Omega$ grows

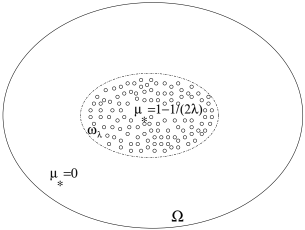


Figure : Vortex patch

- Number of vortices N_ε is proportional to $h_{\rm ex} > \lambda_\Omega |\log \varepsilon|$
- Valid as long as $h_{\rm ex} \ll \frac{1}{\varepsilon^2}$

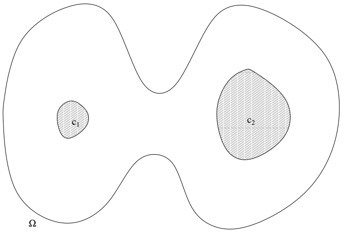
Mean-field model for non minimizing solutions

If u_ε is a solution to the gauged GL equations and $h_{\rm ex} \gg 1$, then $\mu_\varepsilon / h_{\rm ex} \to \mu$ solution to

$$\mu \nabla h = 0 \qquad h = (-\Delta + I)^{-1} \mu$$

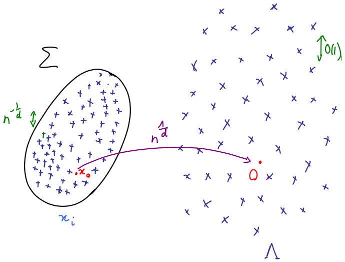
in a suitable weak sense ($\simeq$ Delort)
[Sandier-S '04]

$\rightsquigarrow$ h is constant on the support of μ



Microscopic behavior for energy minimizers

Needs to look at the next order in the energy expansion, and blow-up configuration by factor $\sqrt{N_\varepsilon}$ or $\sqrt{h_{\rm ex}}$.



- After blow-up near a.e. point in ω_λ, the vortices of a minimizer converge to an infinite discrete point configuration Λ which minimizes $\mathbb{W}(\Lambda)$ where

$$\mathbb{W}(\Lambda) = \min\{\mathcal{W}(H), -\Delta H = 2\pi \sum_{p\in\Lambda} \delta_p - 1\}$$

$$\mathcal{W}(H) = \lim_{R\to\infty} \frac{1}{|B_R|} \lim_{\eta\to 0} \Big(\int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi_{B_R} |\nabla H|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi_{B_R}(p) \Big)$$

[Sandier-S '12]

- infinite configuration analogue of the BBH renormalized energy
- total logarithmic interaction of an infinite system of point charges, neutralized by a uniform negative background
- For periodic configuration there is a more explicit expression:

$$\mathbb{W}(\Lambda) = C_1 \sum_{i\neq j} G(a_i - a_j) + C_2$$

G= periodic Green function

- Among Bravais lattices ($=A\mathbb{Z}^2$) of volume 1, $\mathbb{W}$ is uniquely minimized by the triangular lattice ($=60^o$)

- After blow-up near a.e. point in ω_λ, the vortices of a minimizer converge to an infinite discrete point configuration Λ which minimizes $\mathbb{W}(\Lambda)$ where

$$\mathbb{W}(\Lambda) = \min\{\mathcal{W}(H), -\Delta H = 2\pi \sum_{p\in\Lambda} \delta_p - 1\}$$

$$\mathcal{W}(H) = \lim_{R\to\infty} \frac{1}{|B_R|} \lim_{\eta\to 0} \left(\int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi_{B_R} |\nabla H|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi_{B_R}(p) \right)$$

[Sandier-S '12]

- infinite configuration analogue of the BBH renormalized energy
- total logarithmic interaction of an infinite system of point charges, neutralized by a uniform negative background
- For periodic configuration there is a more explicit expression:

$$\mathbb{W}(\Lambda) = C_1 \sum_{i\neq j} G(a_i - a_j) + C_2$$

$G =$ periodic Green function

- Among Bravais lattices ($= A\mathbb{Z}^2$) of volume 1, $\mathbb{W}$ is uniquely minimized by the triangular lattice ($= 60^o$)

- After blow-up near a.e. point in ω_λ, the vortices of a minimizer converge to an infinite discrete point configuration Λ which minimizes $\mathbb{W}(\Lambda)$ where

$$\mathbb{W}(\Lambda) = \min\{\mathcal{W}(H), -\Delta H = 2\pi \sum_{p\in\Lambda} \delta_p - 1\}$$

$$\mathcal{W}(H) = \lim_{R\to\infty} \frac{1}{|B_R|} \lim_{\eta\to 0} \left(\int_{\mathbb{R}^2 \setminus \cup_{p\in\Lambda} B(p,\eta)} \chi_{B_R} |\nabla H|^2 + \pi \log \eta \sum_{p\in\Lambda} \chi_{B_R}(p) \right)$$

[Sandier-S '12]

- infinite configuration analogue of the BBH renormalized energy
- total logarithmic interaction of an infinite system of point charges, neutralized by a uniform negative background
- For periodic configuration there is a more explicit expression:

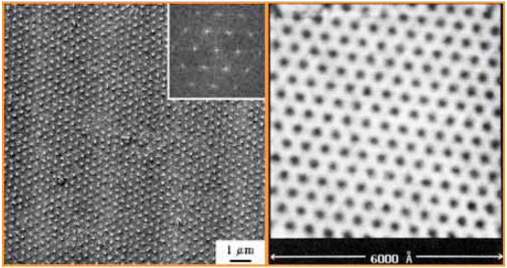
$$\mathbb{W}(\Lambda) = C_1 \sum_{i\neq j} G(a_i - a_j) + C_2$$

G= periodic Green function

- Among Bravais lattices ($=A\mathbb{Z}^2$) of volume 1, $\mathbb{W}$ is uniquely minimized by the triangular lattice ($=60^o$)

The Abrikosov lattice

Compare with Abrikosov lattice seen in experiments in superconductors:



$\rightsquigarrow$ conjecture: the triangular lattice achieves the global minimum of $\mathbb{W}$

Dynamics in the case $N_\varepsilon \gg 1$

Back to

$$\boxed{\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(PGL)}$$

$$\boxed{iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \operatorname{div}(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \tag{1}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '94]: if $\mu \geq 0$

$$\partial_t \mu - \operatorname{div}(\mu \nabla h) = 0 \qquad h = -\Delta^{-1}\mu \tag{2}$$

formally the gradient flow of $F(\mu) = \frac{1}{2}\int |\nabla \Delta^{-1}\mu|^2$ for the 2-Wasserstein metric (à la [Otto, Ambrosio-Gigli-Savaré])

Dynamics in the case $N_\varepsilon \gg 1$

Back to

$$\boxed{\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(PGL)}$$

$$\boxed{iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2} \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \mathrm{div}\,(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \tag{1}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '94]: if $\mu \geq 0$

$$\partial_t \mu - \mathrm{div}\,(\mu \nabla h) = 0 \qquad h = -\Delta^{-1}\mu \tag{2}$$

formally the gradient flow of $F(\mu) = \frac{1}{2}\int |\nabla \Delta^{-1}\mu|^2$ for the 2-Wasserstein metric (à la [Otto, Ambrosio-Gigli-Savaré])

- Study of (2): existence of weak solutions (weak notion à la Delort), uniqueness in the class L^∞, gradient flow approach, asymptotic self-similar profile
$$\mu(t) = \frac{1}{\pi t}\mathbf{1}_{B_{\sqrt{t}}}$$
[Lin-Zhang '00, Du-Zhang '03, Ambrosio-S '08, S-Vazquez '13]

Rigorous convergence results:

- (PGL) case : [Kurzke-Spirn '14] convergence of $\mu_\varepsilon/(2\pi N_\varepsilon)$ to μ solving (2) under assumption $N_\varepsilon \leq (\log\log\ |\log\varepsilon|)^{1/4}$ + well-preparedness
- (GP) case: [Jerrard-Spirn '15] convergence to μ solving (1) under assumption $N_\varepsilon \leq (\log\ |\log\varepsilon|)^{1/2}$ + well-preparedness
- both proofs "push" the fixed N proof (taking limits in the evolution of the energy density) by making it more quantitative
- difficult to go beyond these dilute regimes without controlling distance between vortices, possible collisions, etc

Alternative method: the “modulated energy"

- Exploits the regularity and stability of the solution to the limit equation
- Works for dissipative as well as conservative equations
- Works for gauged model as well

Let $\mathrm{v}(t)$ be the expected limiting velocity field (such that $\langle \nabla u_\varepsilon, iu_\varepsilon \rangle \rightharpoonup \mathrm{v}$ and $\mathrm{curl}\,\mathrm{v} = \mu$). Define the modulated energy

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2},$$

modelled on the Ginzburg-Landau energy.

Analogy with “modulated entropy" methods in kinetic to fluid limits.

Alternative method: the "modulated energy"

- Exploits the regularity and stability of the solution to the limit equation
- Works for dissipative as well as conservative equations
- Works for gauged model as well

Let $\mathrm{v}(t)$ be the expected limiting velocity field (such that $\langle \nabla u_\varepsilon, iu_\varepsilon \rangle \rightharpoonup \mathrm{v}$ and $\mathrm{curl}\,\mathrm{v} = \mu$). Define the modulated energy

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2},$$

modelled on the Ginzburg-Landau energy.
Analogy with "modulated entropy" methods in kinetic to fluid limits.

Main result: Gross-Pitaevskii case

Theorem (S. '15)

Assume u_ε solves (GP) and let N_ε be such that $|\log\varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$. Let v be a $L^\infty(\mathbb{R}_+, C^{0,1})$ solution to the incompressible Euler equation

$$\boxed{\begin{cases} \partial_t \mathrm{v} = 2\mathrm{v}^\perp \mathrm{curl}\,\mathrm{v} + \nabla p & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases}}$$

with $\mathrm{curl}\,\mathrm{v} \in L^\infty(L^1)$.
Let $\{u_\varepsilon\}_{\varepsilon>0}$ be solutions associated to initial conditions u_ε^0, with $|u_\varepsilon^0| \le 1$ and $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le o(N_\varepsilon^2)$. Then, for every $t \ge 0$, we have

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \to \mathrm{v} \quad \text{in } L^2(\mathbb{R}^2).$$

Implies of course the convergence of the vorticity $\mu_\varepsilon / N_\varepsilon \to \mathrm{curl}\,\mathrm{v}$

Main result: parabolic case

Theorem (S. '15)

Assume u_ε *solves (PGL) and let* N_ε *be such that* $1 \ll N_\varepsilon \leq O(|\log \varepsilon|)$*. Let* v *be a* $L^\infty([0,T], C^{1,\gamma})$ *solution to*

- *if* $N_\varepsilon \ll |\log \varepsilon|$

$$\boxed{\begin{cases} \partial_t \mathrm{v} = -2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} + \nabla p & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases}}$$

- *if* $N_\varepsilon \sim \lambda |\log \varepsilon|$

$$\boxed{\partial_t \mathrm{v} = \frac{1}{\lambda}\nabla \mathrm{div}\ \mathrm{v} - 2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} \quad \text{in } \mathbb{R}^2.}$$

Then under same assumptions on initial data, for every $t \leq T$ *we have*

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \to \mathrm{v} \quad \text{in } L^p_{loc}(\mathbb{R}^2), \ p < 2.$$

Taking the curl of the equation yields back the Chapman-Rubinstein-Schatzman-E equation (2) if $N_\varepsilon \ll |\log \varepsilon|$, but *not* if $N_\varepsilon \propto |\log \varepsilon|$!

Proof method and difficulties

- Go around the question of minimal vortex distances by using instead the modulated energy and showing a Gronwall inequality

$$\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t)) \leq C(\mathcal{E}_\varepsilon(u_\varepsilon(t)) - \pi N_\varepsilon|\log\varepsilon|) + o(N_\varepsilon^2)$$

- In the parabolic case, it requires removing the energy concentrating in the vortices, control by "ball construction" method, and a control on the vortex velocities ("product estimate")
- Relies on algebraic simplifications in computing $\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t))$ which reveal only quadratic terms
- Uses the regularity of v to bound corresponding terms
- Insight is to think of v as a spatial gauge vector and $\mathrm{div}\ \mathrm{v}$ (resp. p) as a temporal gauge

THANK YOU FOR YOUR ATTENTION !