# Complex dynamics, bifurcations, and arithmetic

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# Complex dynamics and arithmetic geometry

Complex/Algebraic
Dynamical Systems

Arithmetic Geometry

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Dynamical Systems

Arithmetic Geometry

$X = \text{algebraic variety}/\mathbb{C}$

$f : X \to X$

Study orbits of points

$x, f(x), f^2(x), f^3(x), \ldots$

$X = \mathbb{C}$ and $f = \text{polynomial}$

$X = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$, a torus, with $f(z) = 2z$

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$X =$ algebraic variety$/k$

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Study set of rational points

$X(k)$

$X = \mathbb{A}^1_k$ and $X(k) = k$

$X =$ elliptic curve$/k$, $X(k) = ?$

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dynamics $\longleftrightarrow$ analysis/geometry $\longleftrightarrow$ algebra (static)

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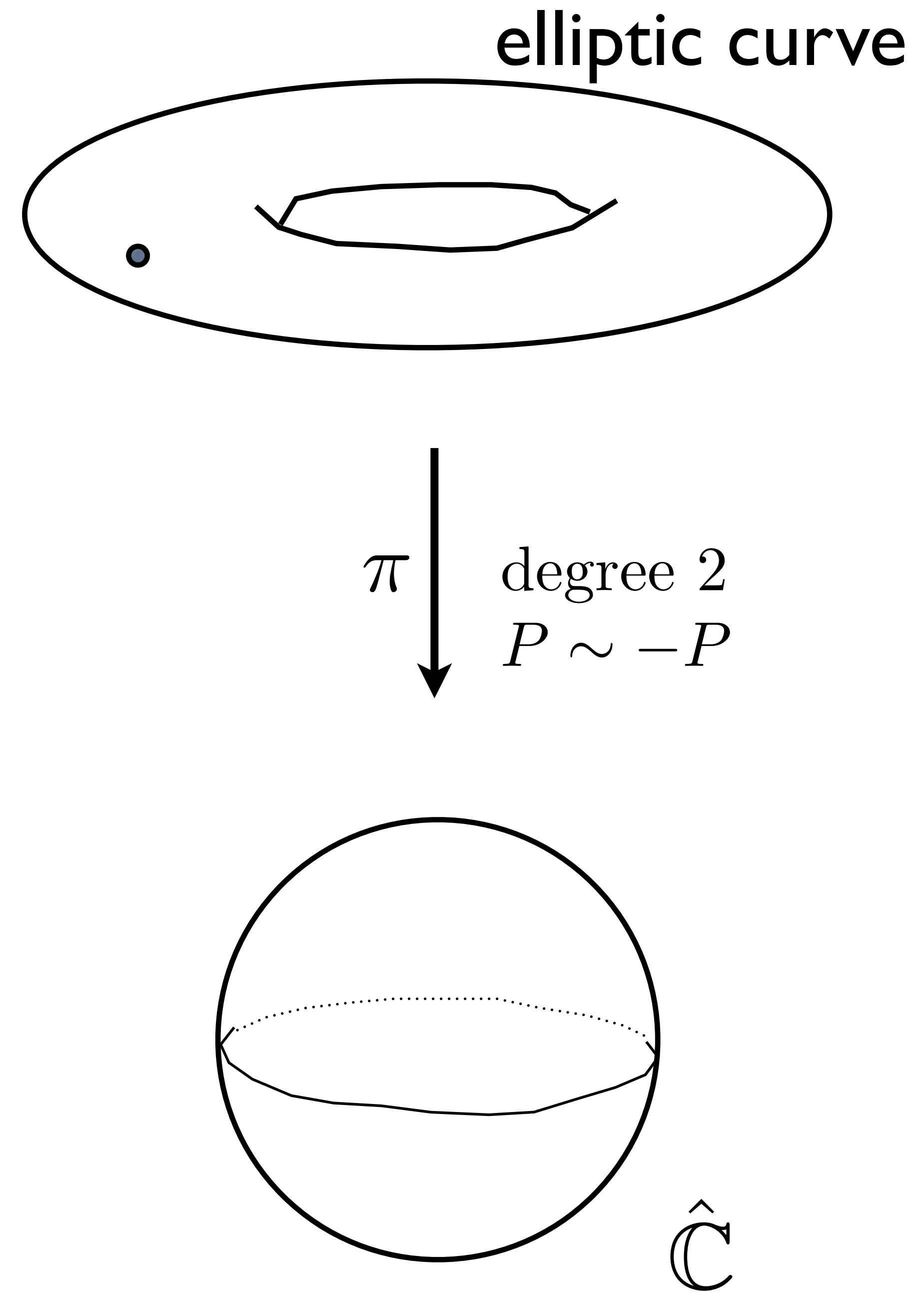
## Special case



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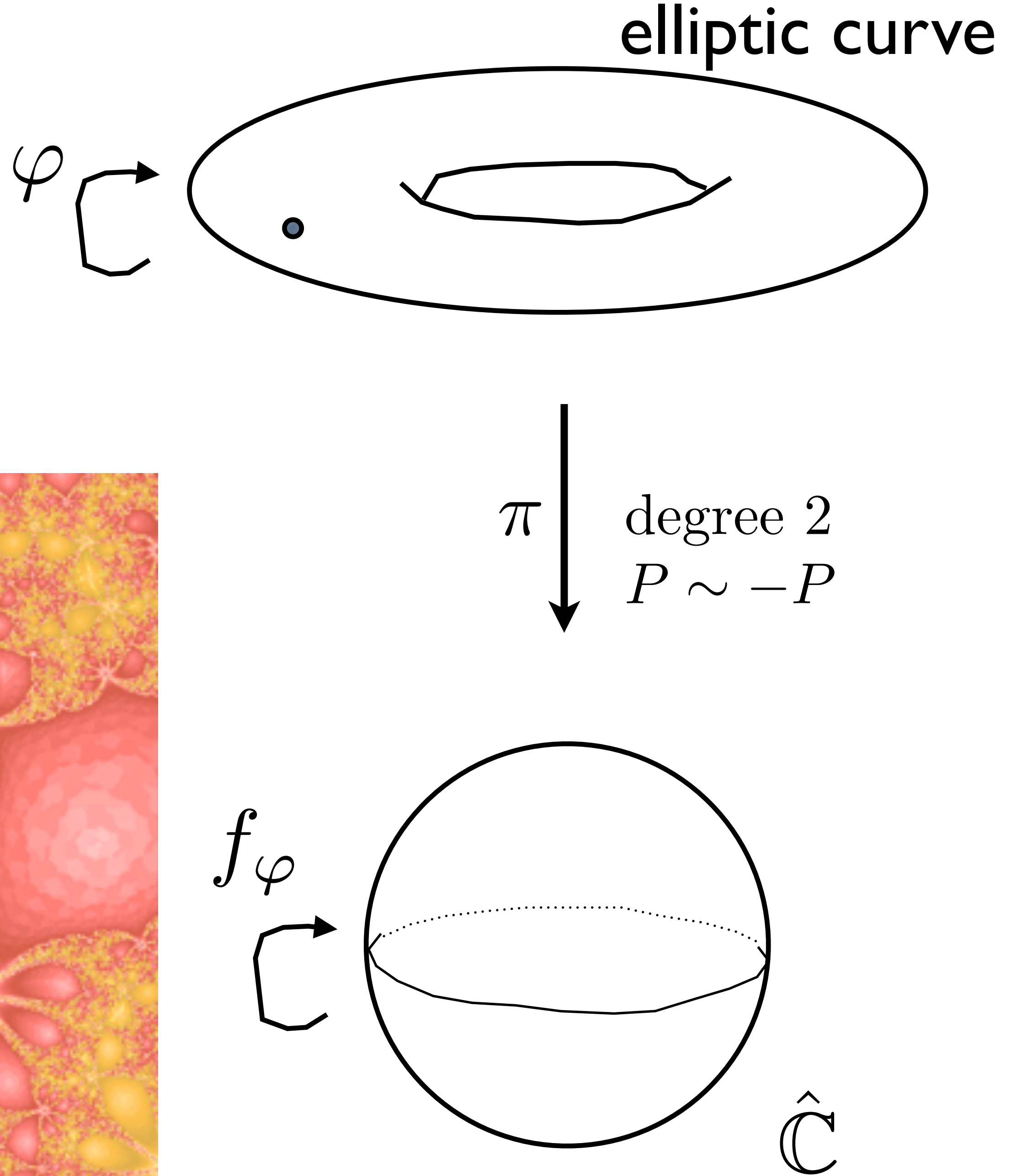

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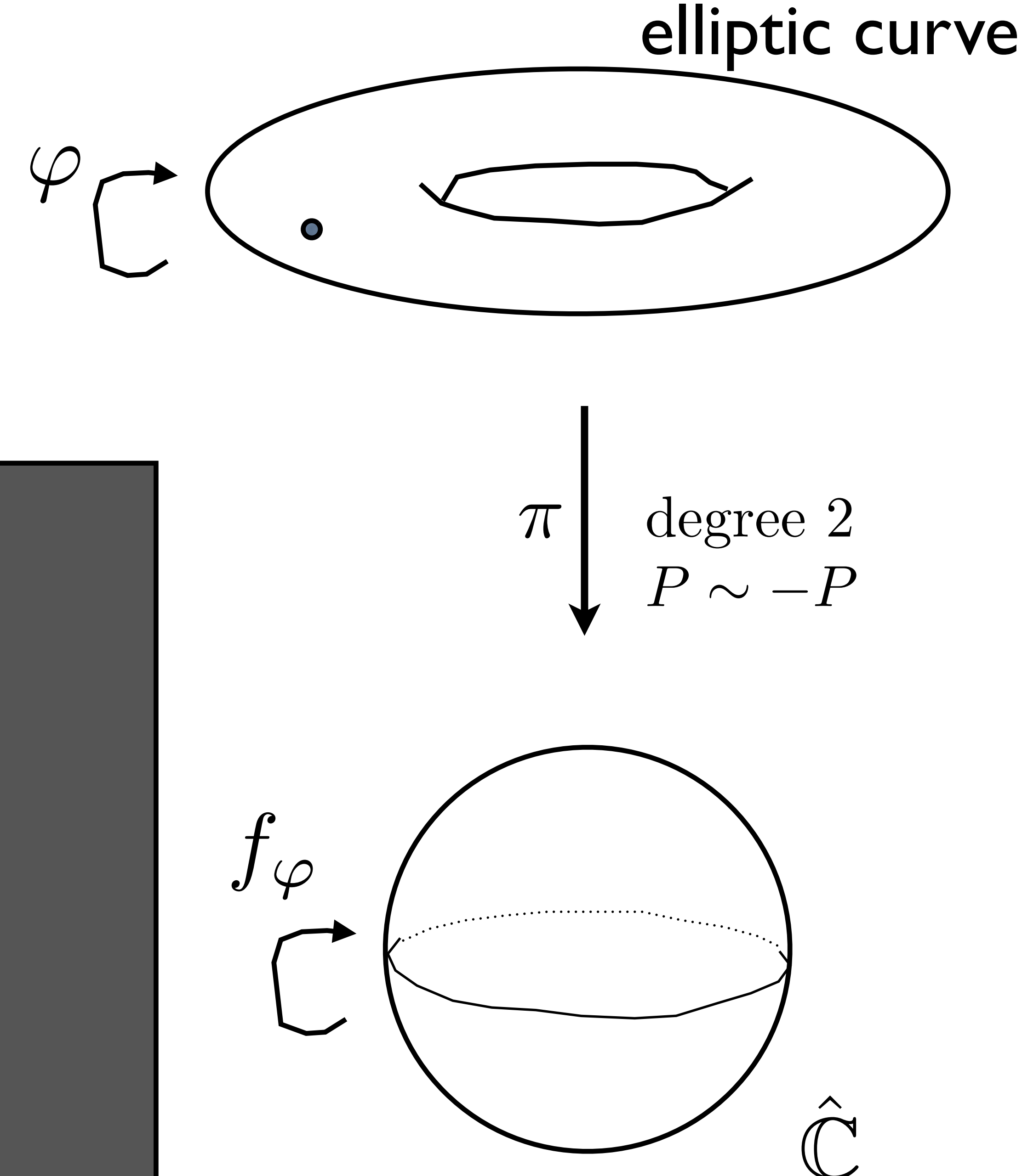

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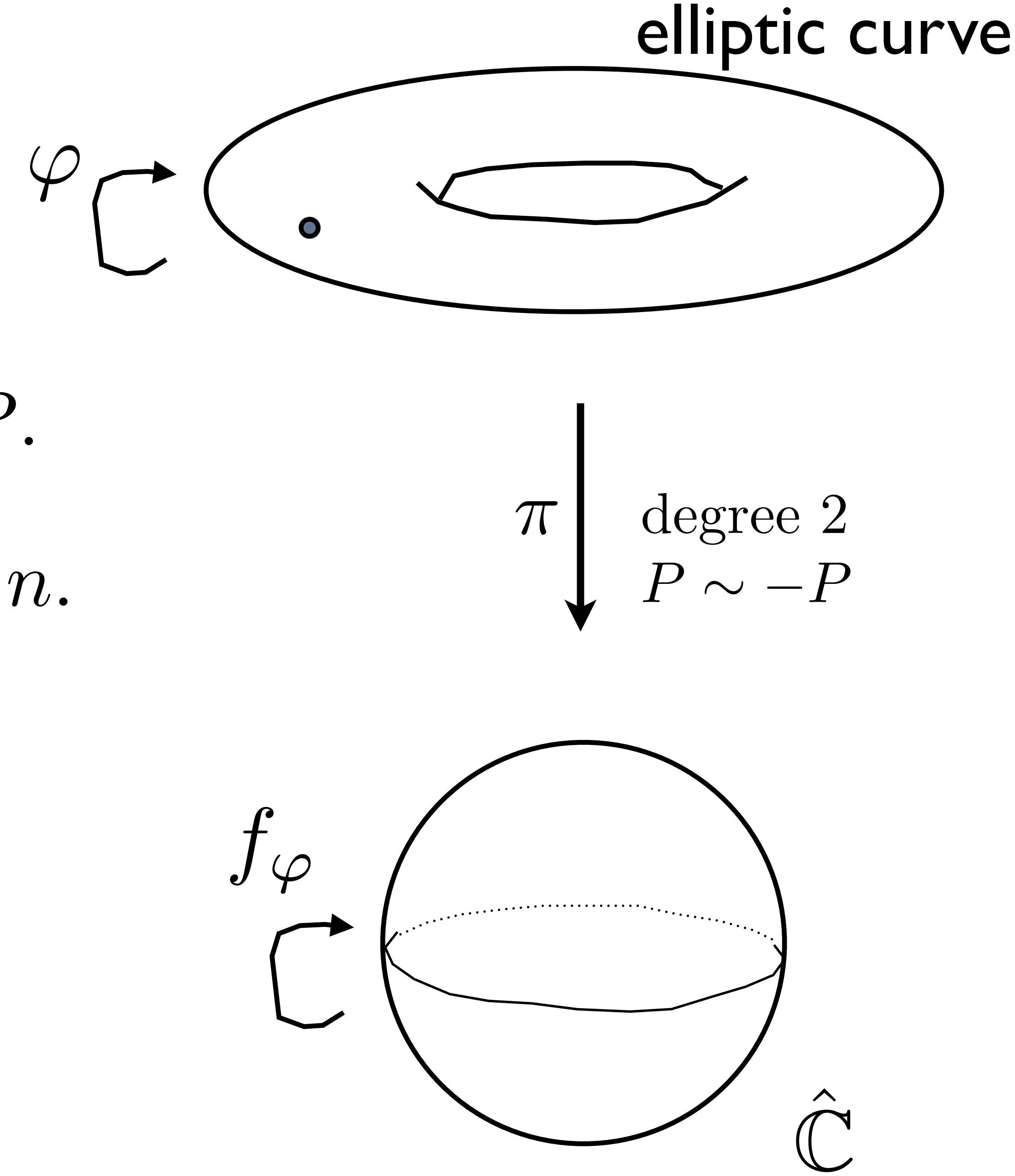


Take, for example, $\varphi(P) = P + P = 2P$.

$P \in E$ is **torsion** if $n \cdot P = 0$ for some $n$.

$P \in E$ is torsion

$\iff$ $P$ is preperiodic for $\varphi$

$\iff$ $\pi(P)$ is preperiodic for $f_\varphi$

# Classical result about elliptic curves

$$E = \text{elliptic curve } /k, \text{ with } k = \text{number field}$$

For example, take $E = \left\{\, y^2 = x(x-1)(x - \frac{22}{37}) \,\right\} \subset \mathbb{P}^2(\mathbb{C})$
with $k = \mathbb{Q}$

**Mordell-Weil Theorem.** (1920s) The set of rational points $E(k)$ forms a finitely-generated group.

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A modern (c. 1960) explanation of this finiteness:

Néron-Tate height function

$$\hat{h}_E : E(\overline{\mathbb{Q}}) \to \mathbb{R}$$

$$\hat{h}_E(P) = \lim_{n\to\infty} \frac{1}{4^n} h_{\mathrm{Weil}}(2^n P)$$

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Key fact: For number fields $k$, the set

$$\{\text{torsion } P \in E(k)\} = \{P \in E(k) : \hat{h}_E(P) = 0\} \text{ is finite.}$$

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What if the field k is a function field?

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View $E/k$ as a complex surface $\mathbf{E} \to X$

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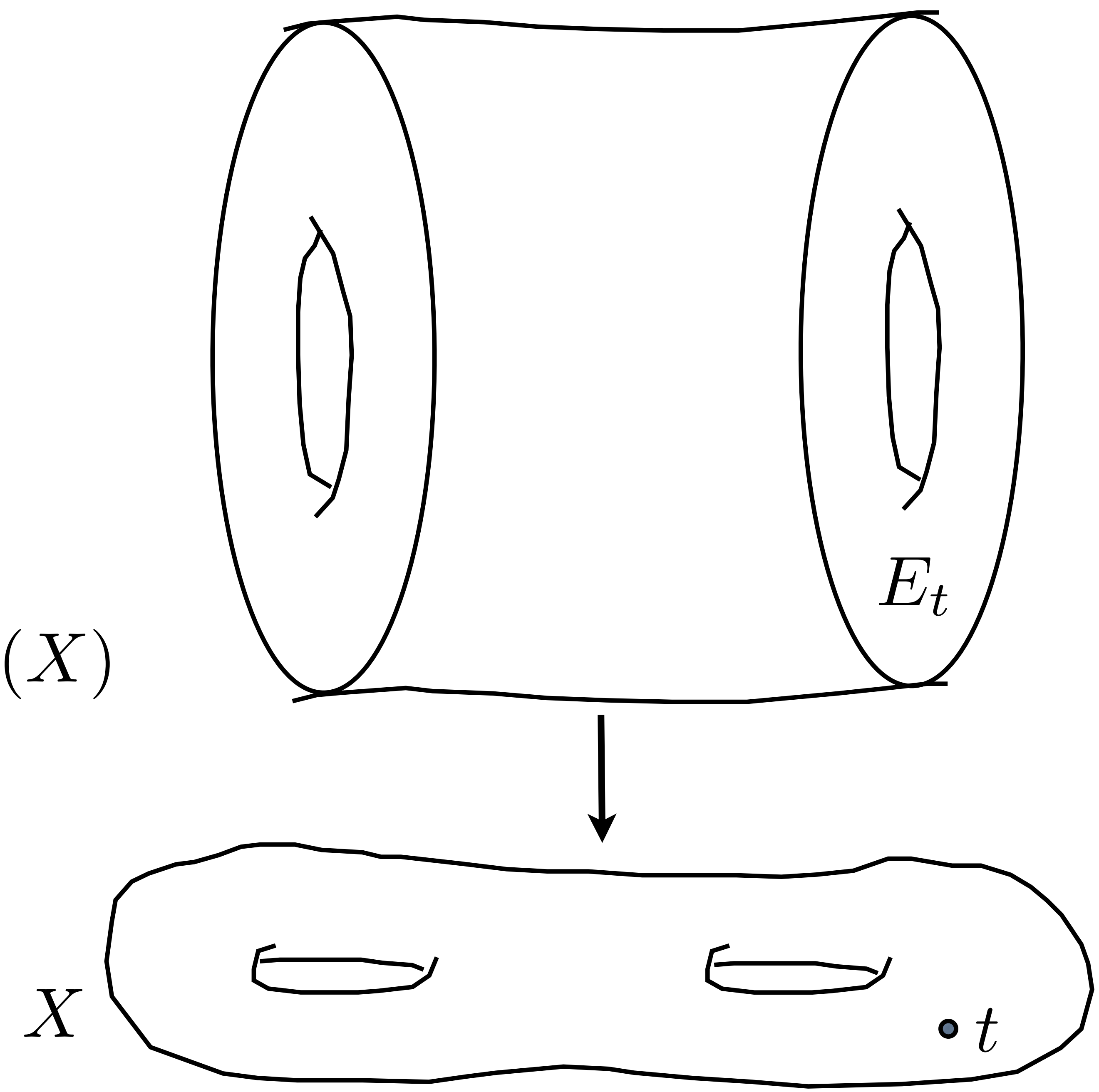

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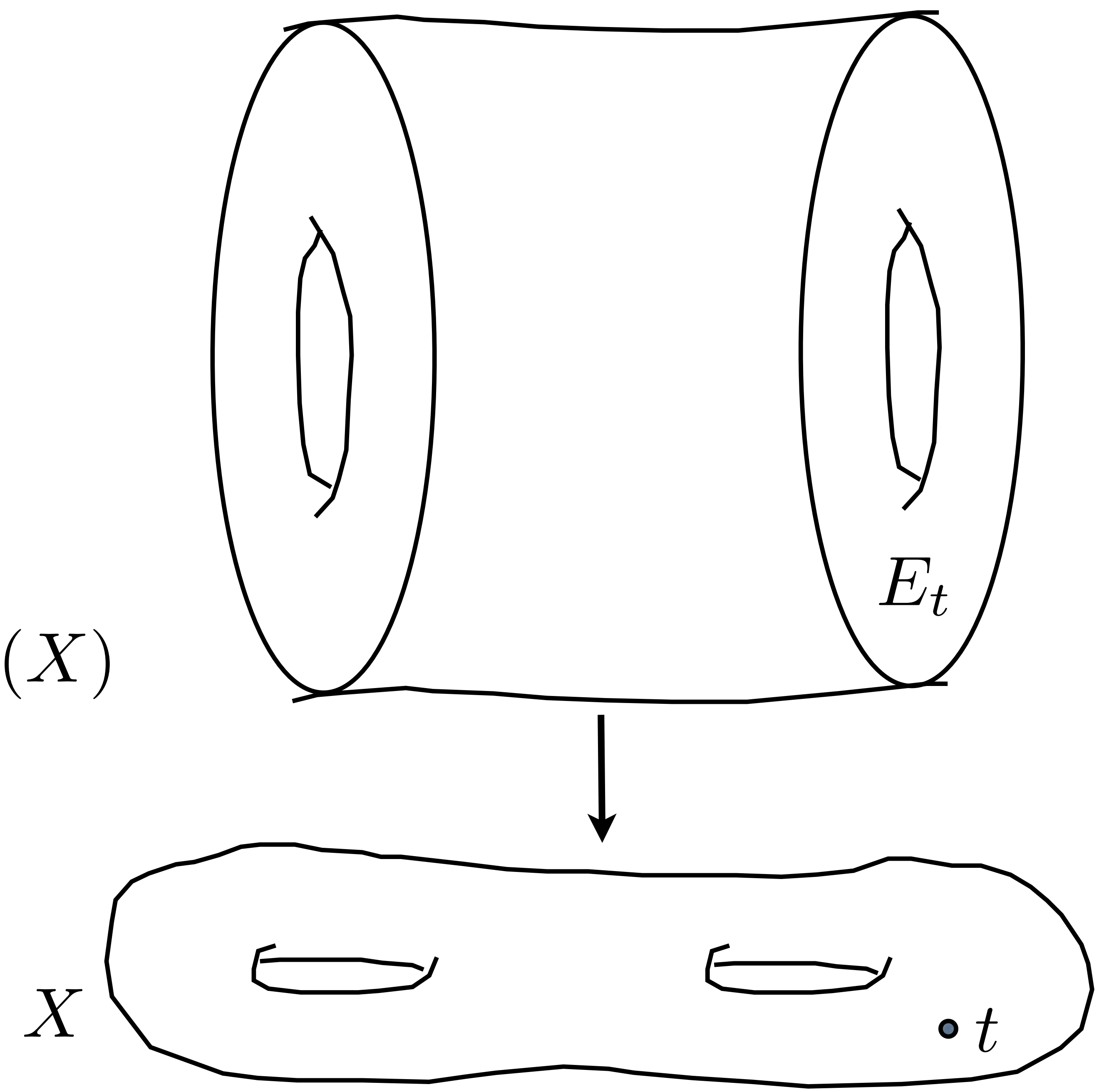


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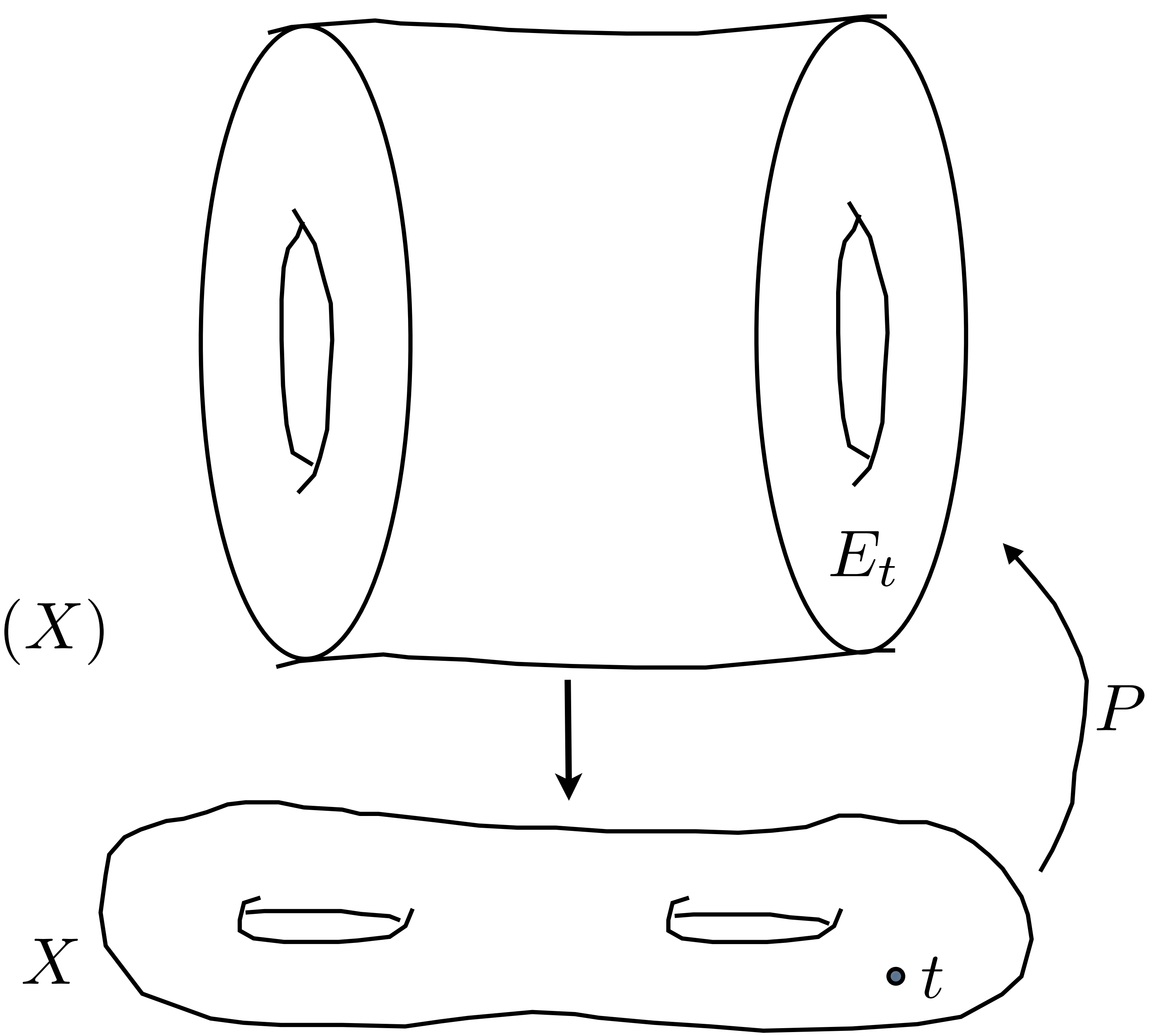


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A rational point is a section from $X$ to the surface $\mathbf{E}$

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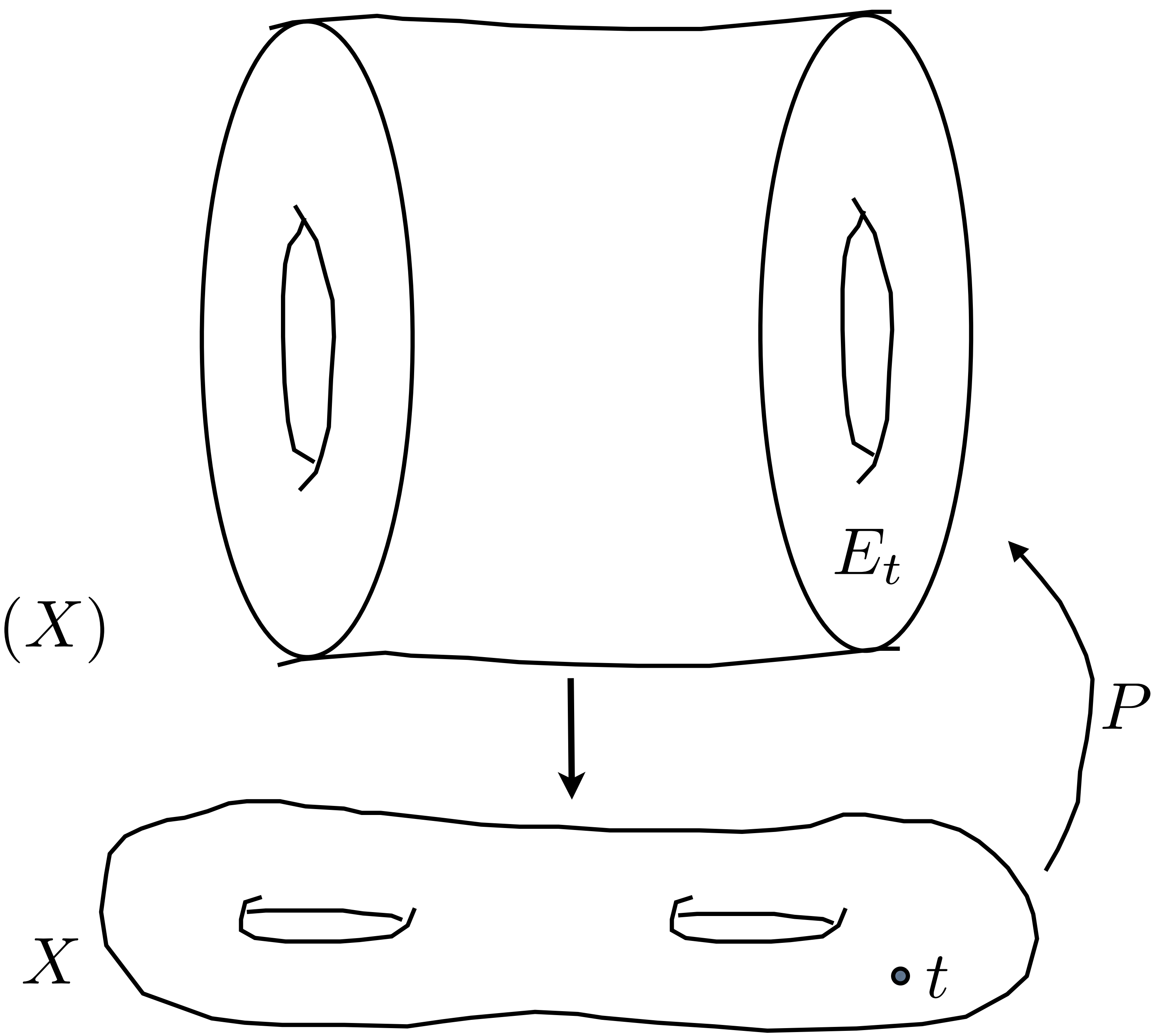

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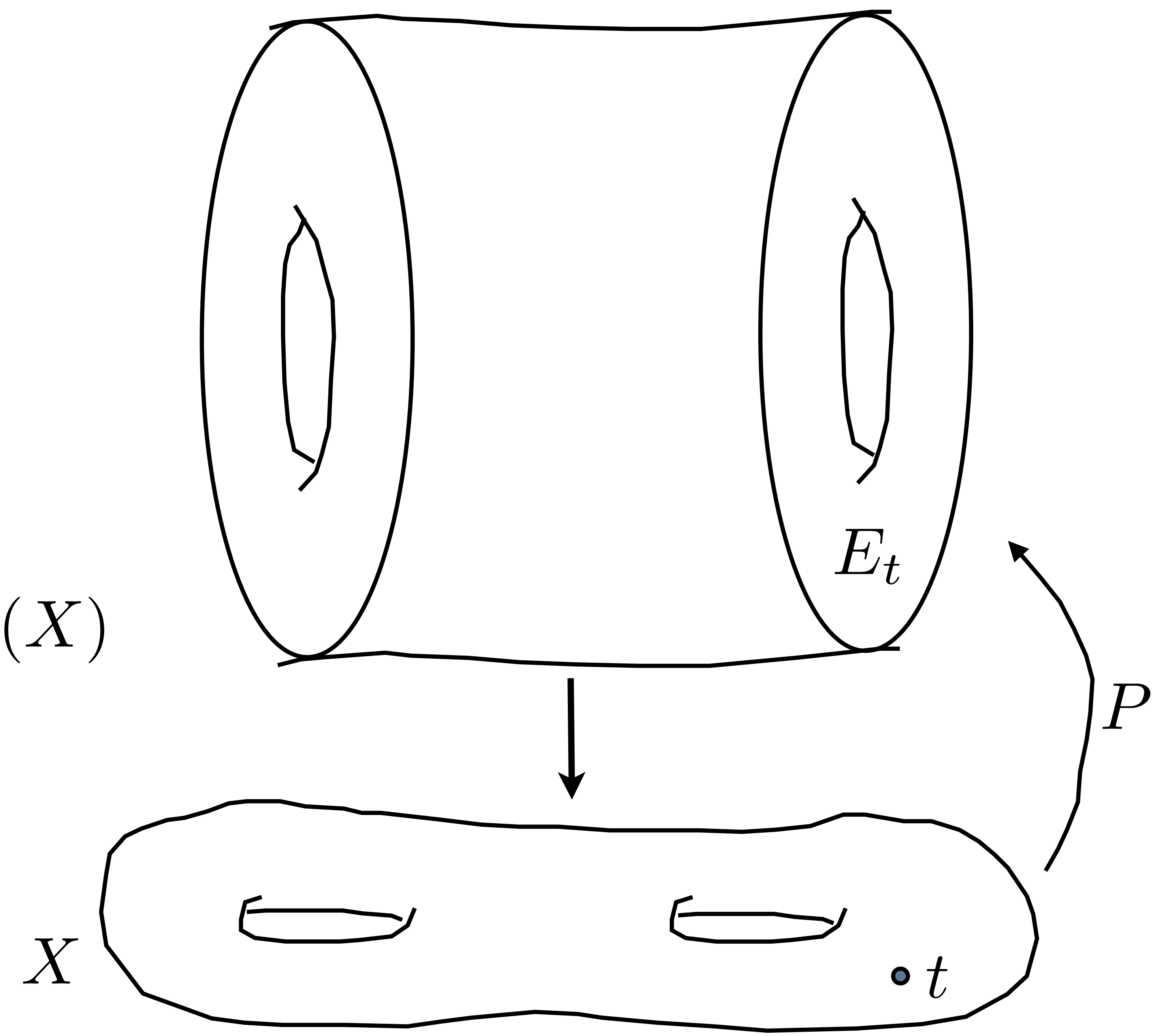

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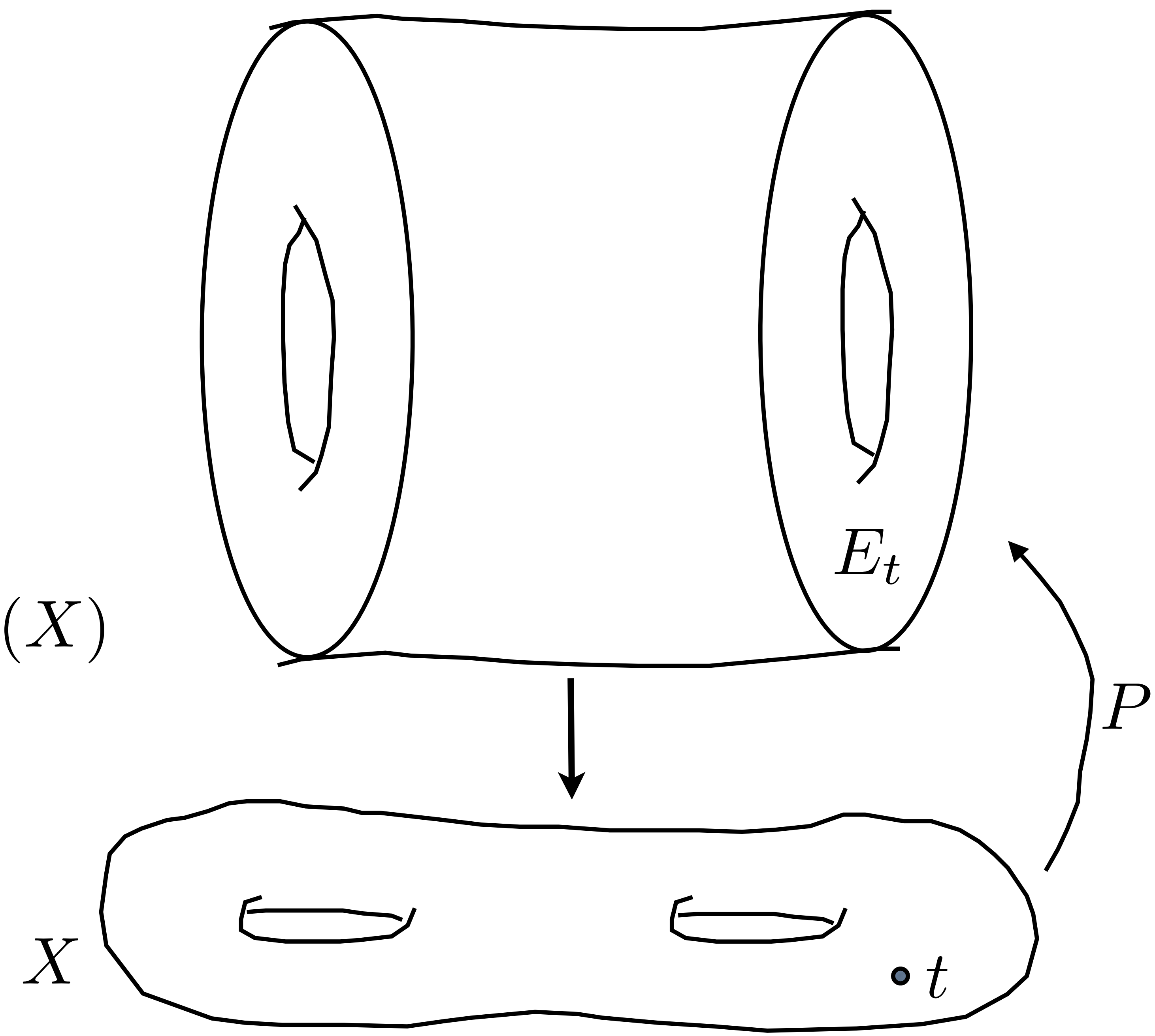

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Proof of finiteness, using dynamics:

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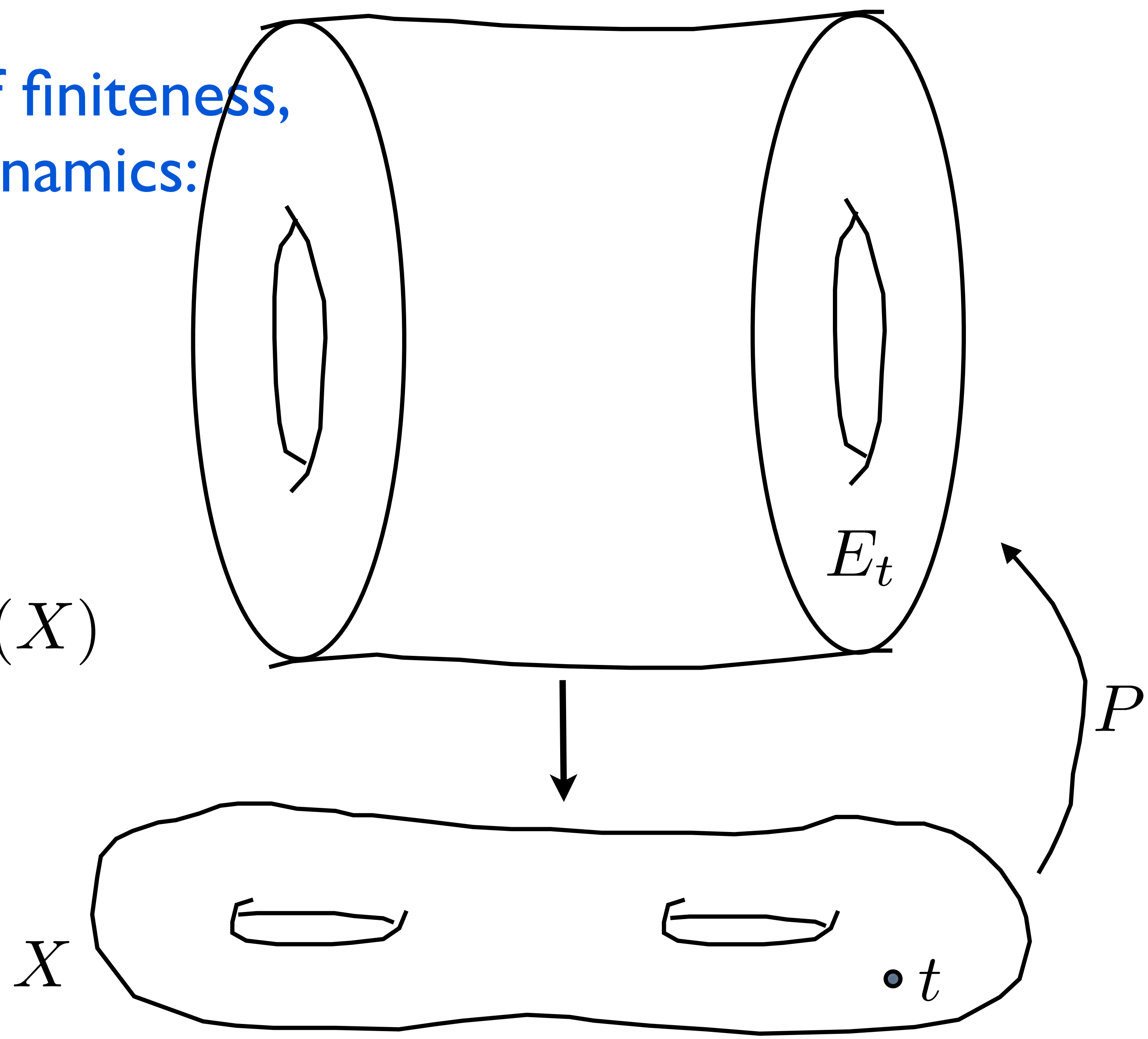

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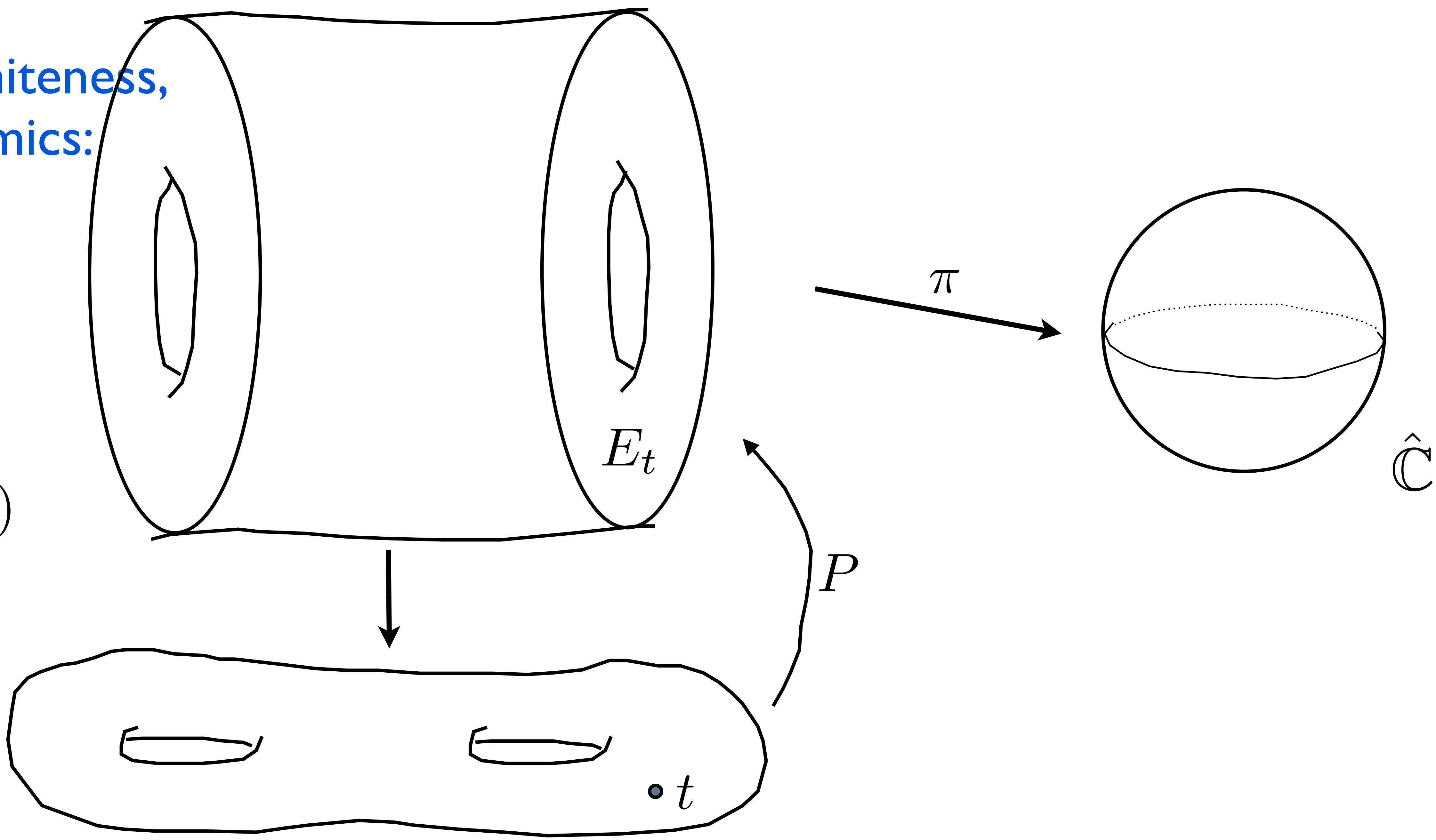

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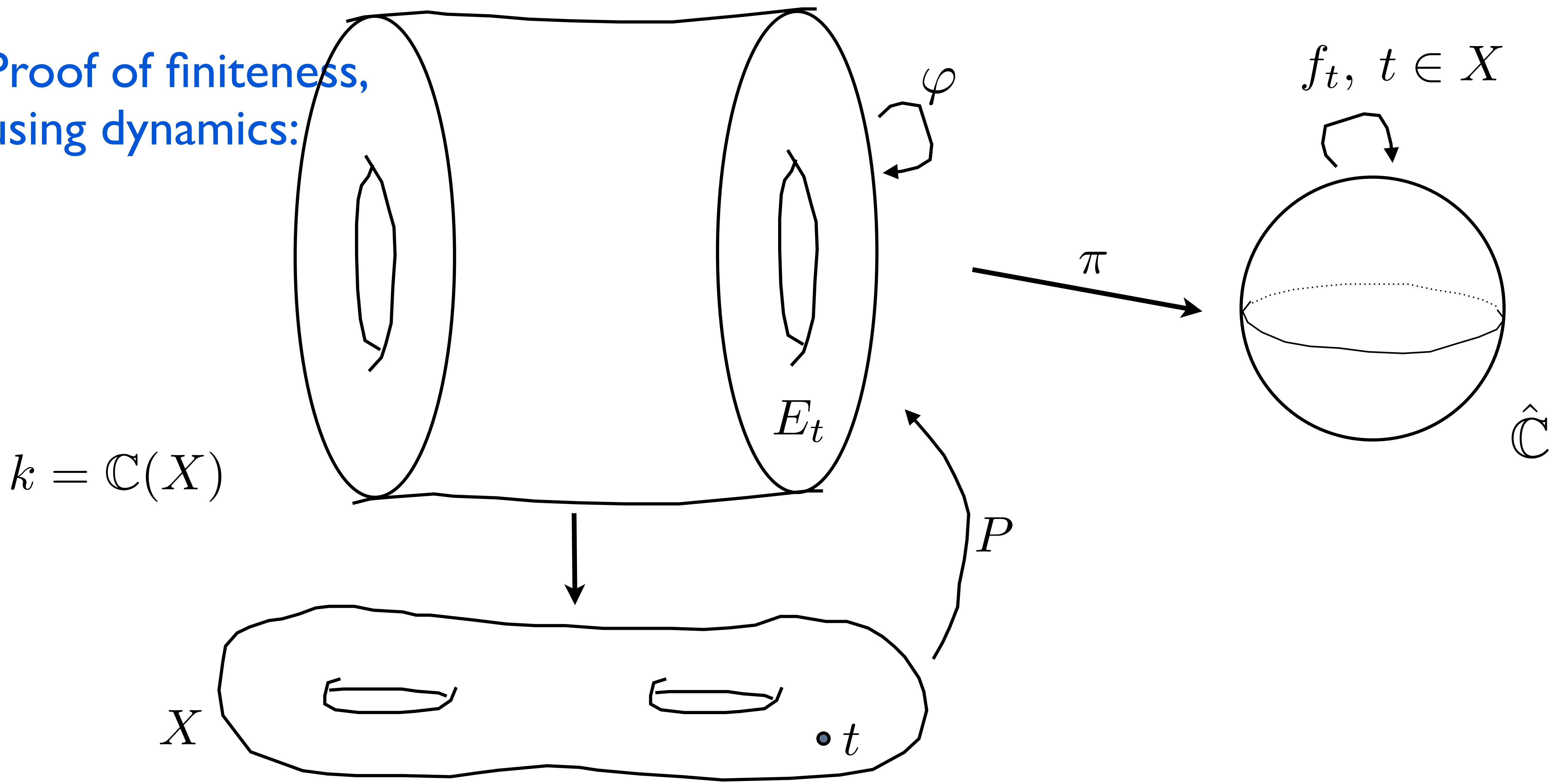

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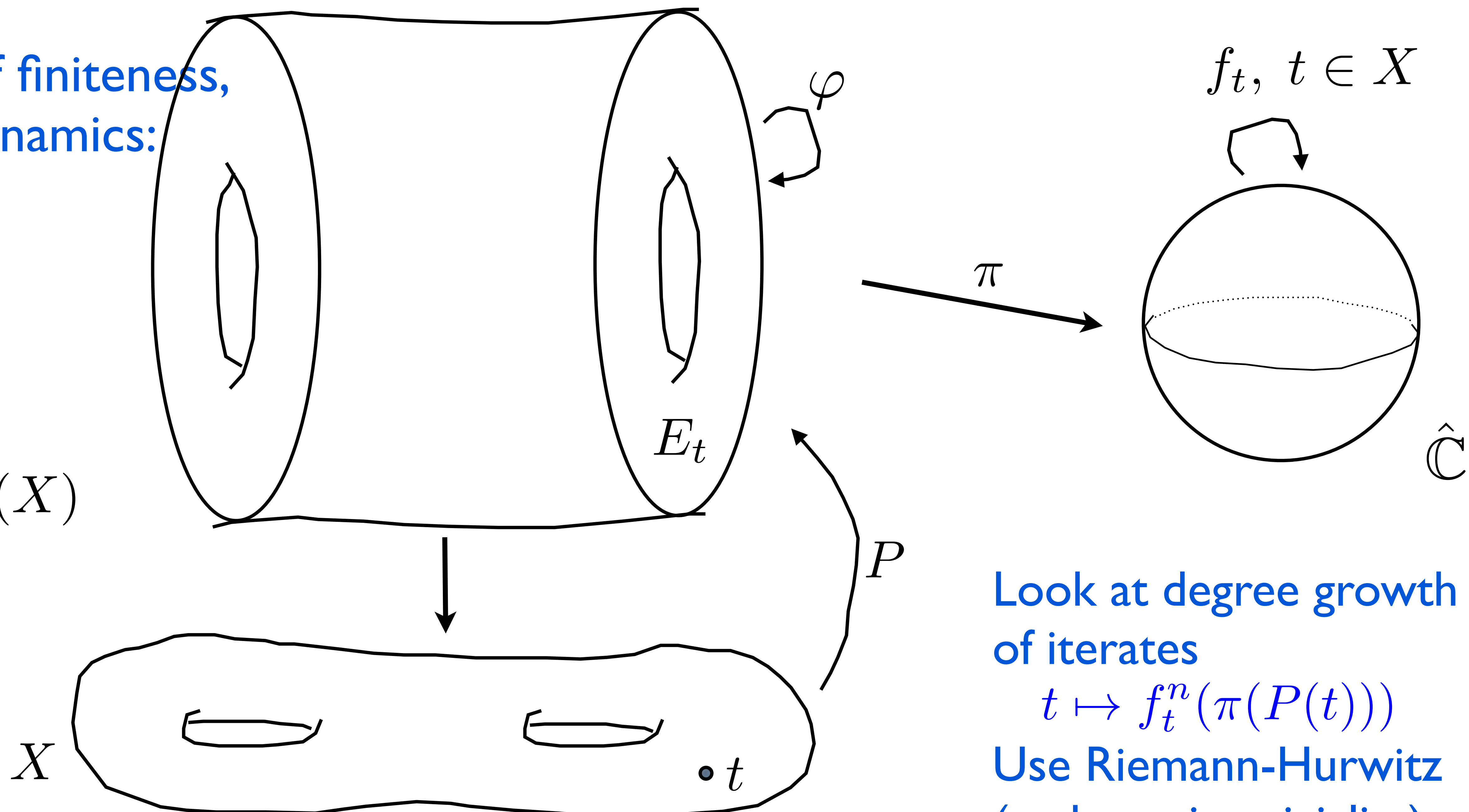


$k = \mathbb{C}(X)$

Look at degree growth of iterates

$t \mapsto f_t^n(\pi(P(t)))$

Use Riemann-Hurwitz (and non-isotriviality).

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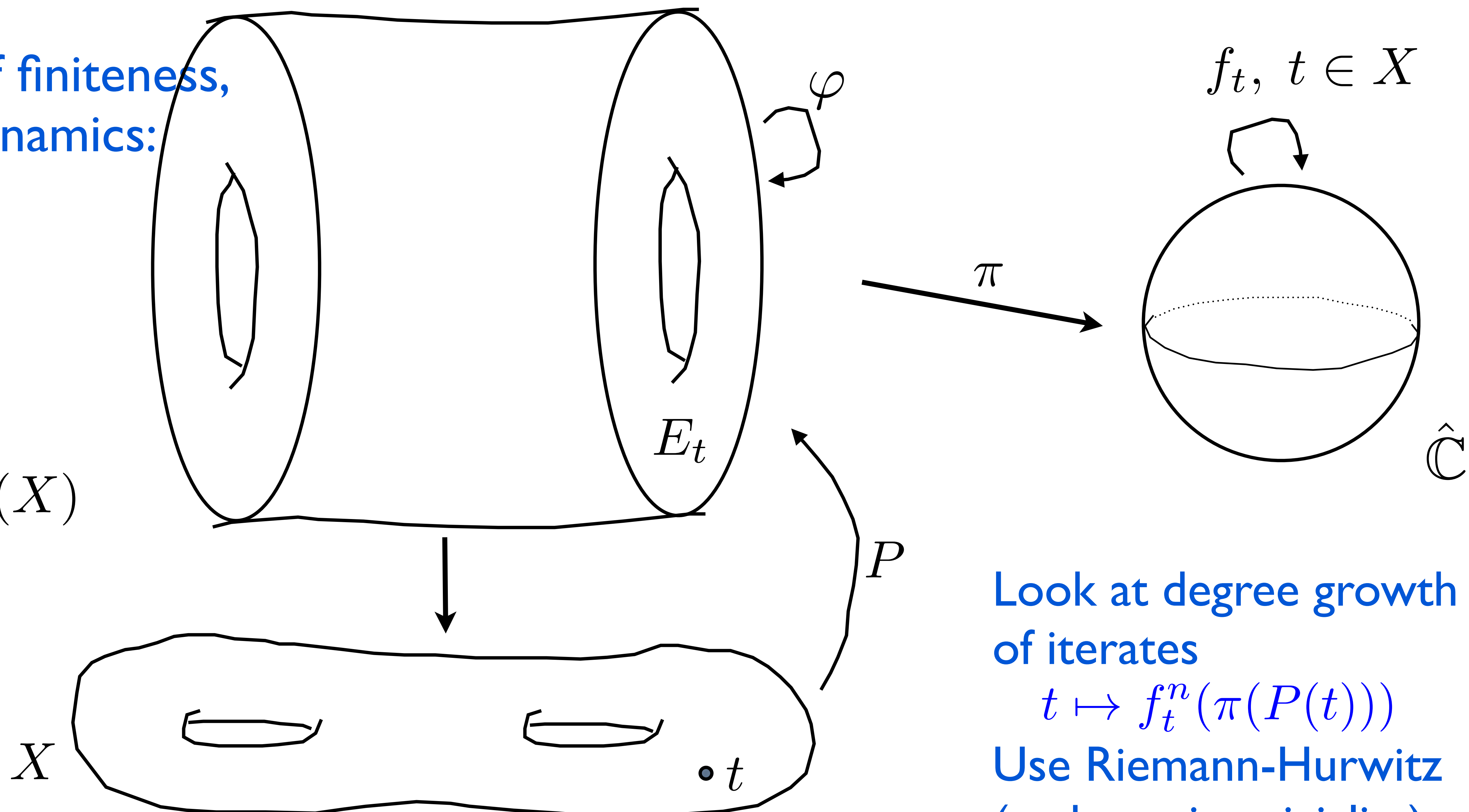


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**For elliptic curves over function fields:** (Lang-Néron 1959, Néron Tate 1960s) If $E/k$ is not isotrivial, then the set of torsion points in $E(k)$ is finite.

$$\Uparrow$$

**For rational functions over function fields:** (Baker 2008) If $f \in k(z)$ is not isotrivial, then the set of preperiodic points in $\mathbb{P}^1(k)$ is finite.

Baker's theorem actually states: (proof uses analysis on Berkovich $\mathbf{P}^1$)

If $f \in k(z)$ is not isotrivial, there exists a constant $B > 0$ so that the set

$$\{P \in \mathbb{P}^1(k) : \hat{h}_f(P) < B\}$$

is finite, where $\hat{h}_f$ is the canonical height on $\mathbb{P}^1(\bar{k})$.

Complex-dynamics proof in (D., 2015). Key ingredients:

non-isotriviality $\Longrightarrow$ bifurcations $\Longrightarrow$ degree growth of $f^n(P)$

Riemann-Hurwitz (topology) $\Longrightarrow$ finiteness

# Dynamical stability and bifurcations: the analytic input

$X$ = Riemann surface
$k = \mathbb{C}(X)$ = meromorphic functions on $X$

$$\begin{matrix} f \in k(z) \\ P \in \mathbb{P}^1(k) \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} f_t, t \in X, \text{ a family of rational functions} \\ P : X \to \hat{\mathbb{C}} \text{ holomorphic} \end{matrix}$$

$(f, P)$ is **stable** if the sequence $\{t \mapsto f_t^n(P(t))\}_n$ is normal on $X$.

Bifurcations can be quantified by a measure, defined locally by

$$U(t) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} \log |f_t^n(P(t))|$$

a subharmonic potential function (written for polynomial f)

$$\mu_P = \Delta U$$

the "bifurcation measure" on X

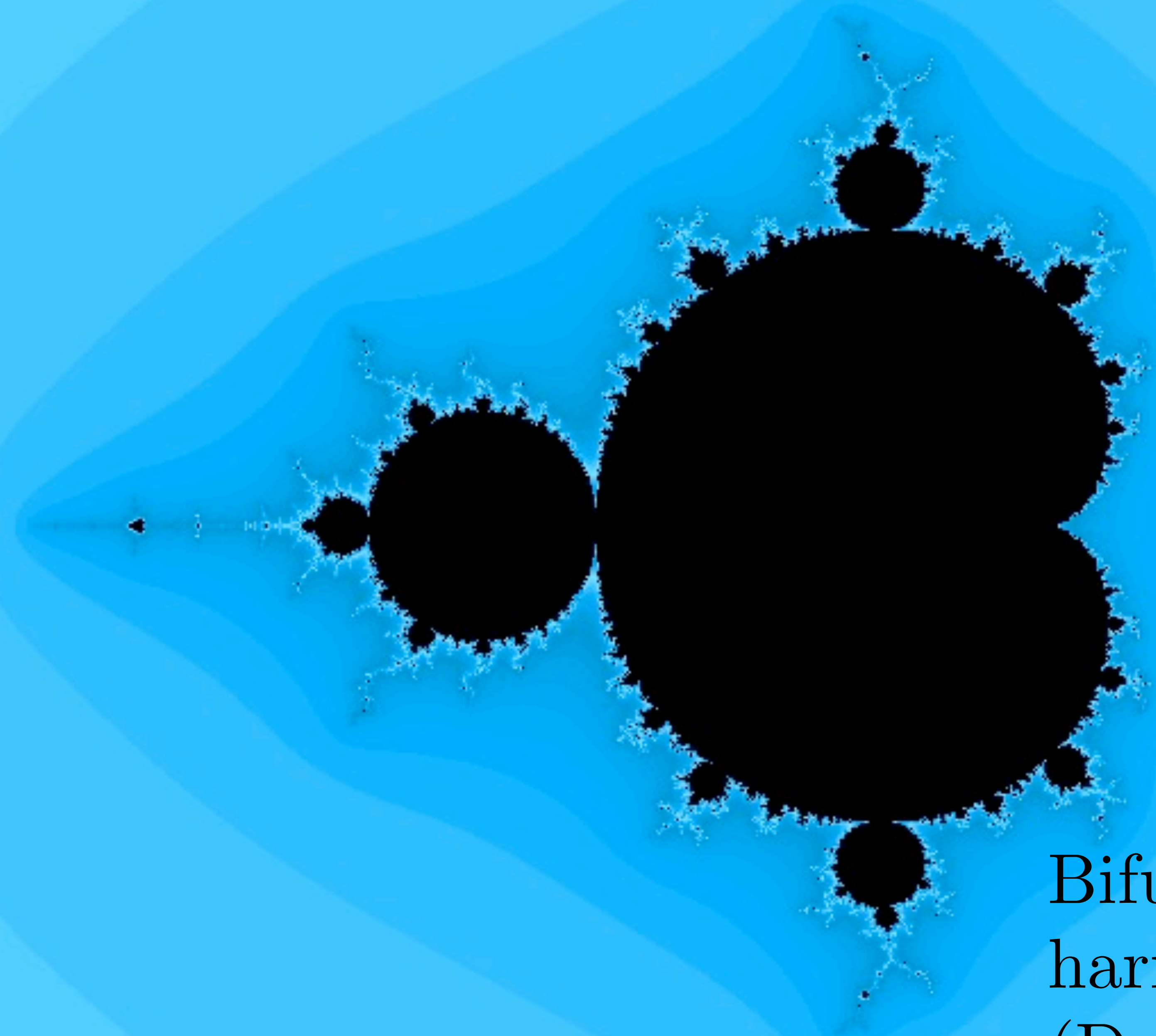
**Theorem.** (D. 2015) If $\mu_P = 0$ on $X$ then $P$ is preperiodic for $f$.

Compare: McMullen (1987), Dujardin-Favre (2008). When P is a critical point of f, this stability coincides with traditional notion of structural stability.

# Example: degree 2 polynomials $\quad f_t(z) = z^2 + t \quad t \in \mathbb{C}$

$$P = 0$$

## The Mandelbrot set

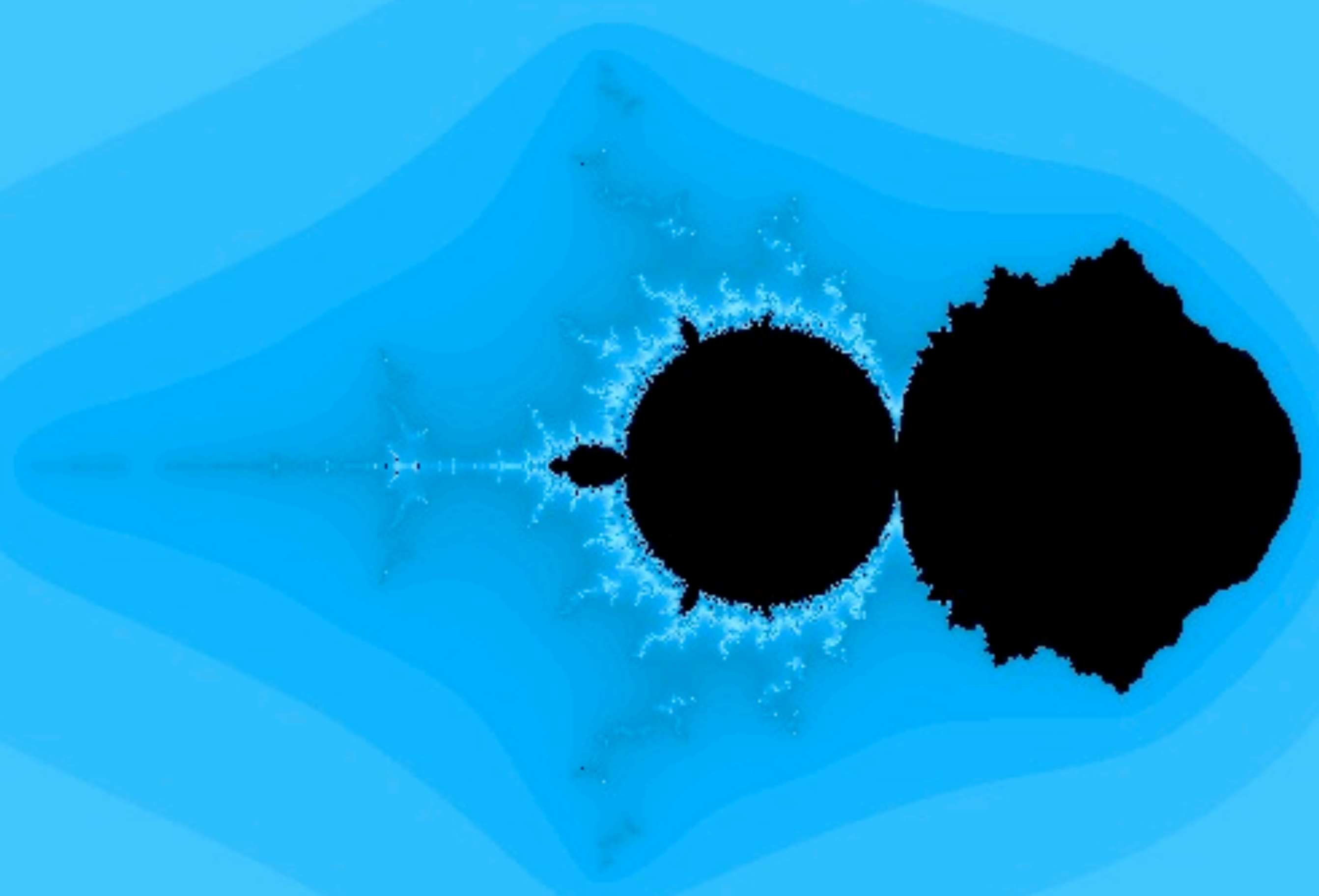


Bifurcation measure $\mu_P$ is harmonic measure on $\partial\mathcal{M}$ (Douady-Hubbard, Sibony 1981, Mañé-Sad-Sullivan 1983)

## Example: degree 2 polynomials $\quad f_t(z) = z^2 + t \qquad t \in \mathbb{C}$

$$P = 1$$

A Mandelbrot-like set



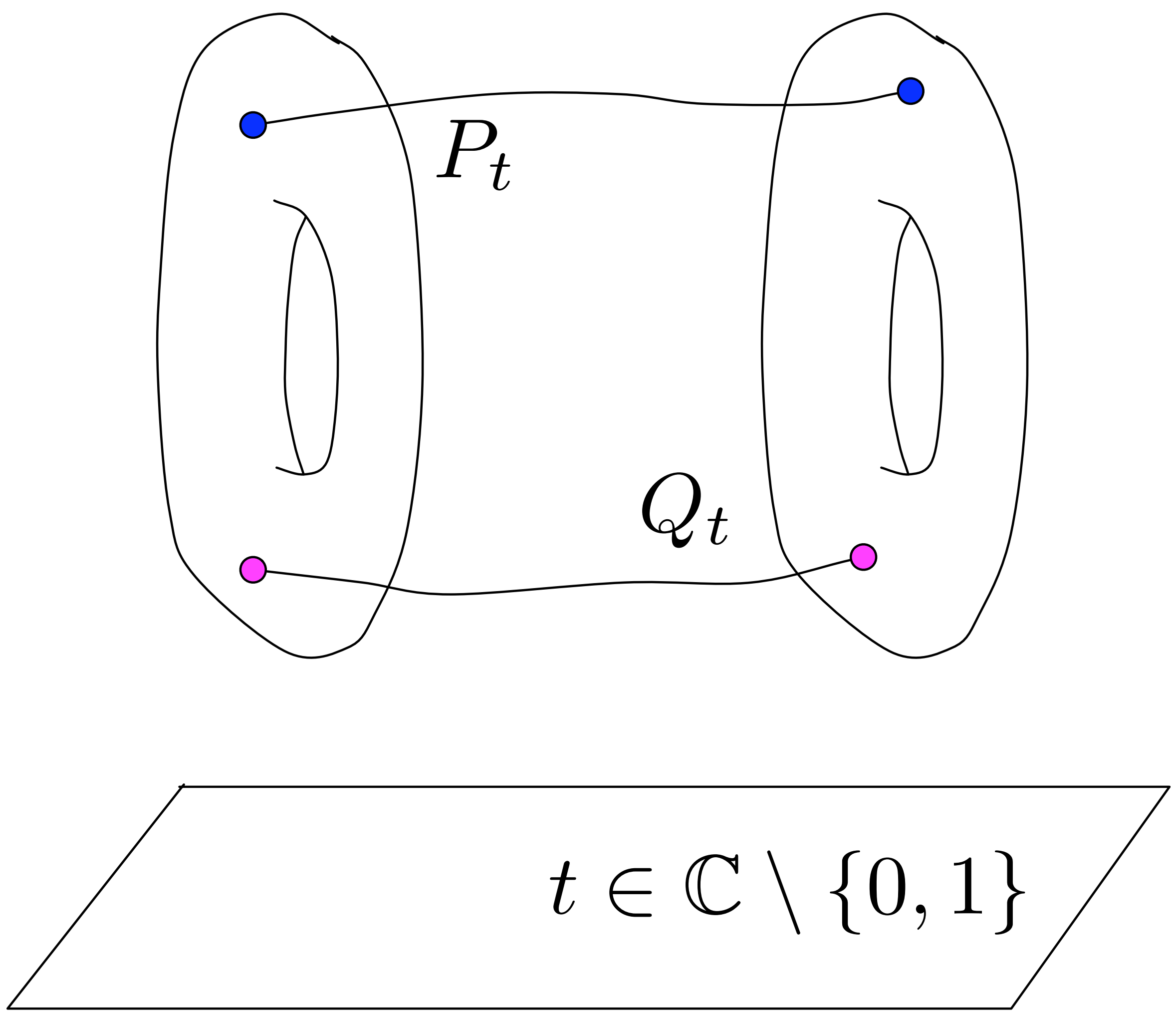
Bifurcation measure $\mu_P$ is harmonic measure on $\partial\mathcal{M}$ (Baker-D. 2014)

# Recent result about elliptic curves

**Theorem.** (Masser-Zannier, 2008–2012, Torsion anomalous points)
Let $E_t$ be the Legendre family of elliptic curves.

$$P_t = (2, \sqrt{2(2-t)}) \quad \text{and} \quad Q_t = (3, \sqrt{6(3-t)})$$

There are only finitely many $t \in \mathbb{C} \setminus \{0,1\}$ for which both $P_t$ and $Q_t$ are torsion points on $E_t$.




$$E_t = \{y^2 = x(x-1)(x-t)\}$$

Compare:
Lang, Manin-Mumford,
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conjectures/theorems

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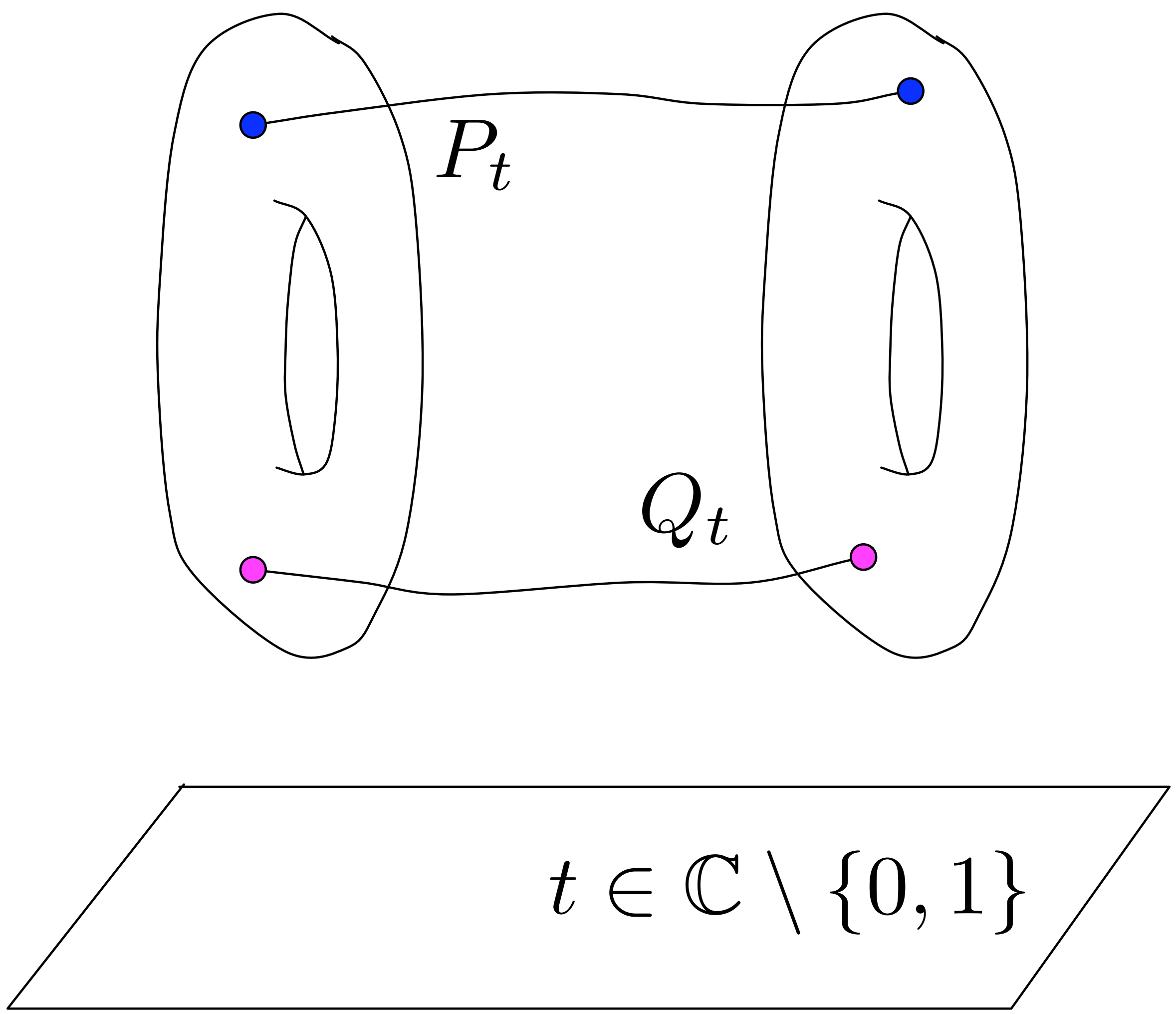
$$P_t = (a, \sqrt{a(a-1)(a-t)}) \text{ and } Q_t = (b, \sqrt{b(b-1)(b-t)})$$

There are only finitely many $t \in \mathbb{C} \setminus \{0,1\}$ for which both $P_t$ and $Q_t$ are torsion points on $E_t$.

$a \neq b \in \mathbb{C} \setminus \{0,1\}$

More generally:
for $a, b \in \overline{\mathbb{C}(t)}$,
assume $nP \neq mQ$ for all
$(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

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# Simultaneously torsion

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$$\Updownarrow$$

**Theorem.** Let

$$f_t(z) = \frac{(z^2-t)^2}{4z(z-1)(z-t)}$$

be the degree-4 Lattès family of rational functions. Fix $a \neq b$ in $\mathbb{C} \setminus \{0,1\}$. Then there are finitely many parameters $t$ for which both $a$ and $b$ are preperiodic.

# Simultaneously preperiodic

# Simultaneously torsion

**Theorem.** (Masser-Zannier, 2008–2012, Torsion anomalous points) Let $E_t$ be the Legendre family of elliptic curves.

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In joint work with Xiaoguang Wang and Hexi Ye, building on my earlier work with Matt Baker, we gave a dynamical proof of this statement. The proof uses both complex dynamics and non-archimedean analysis.

# The stronger, “Bogomolov” version

**Theorem.** Fix $a \neq b$ in $\overline{\mathbb{Q}}$, with $a, b \neq 0, 1$.

$$E_t = \{y^2 = x(x-1)(x-t)\}$$

$$P_t = (a, \sqrt{a(a-1)(a-t)}) \text{ and } Q_t = (b, \sqrt{b(b-1)(b-t)})$$

$$\hat{h}_a(t) := \hat{h}_{E_t}(P_t) = \text{Néron-Tate height}$$

$$\mathrm{Tor}(a) := \{t : P_t \text{ is torsion on } E_t\} = \{t : \hat{h}_a(t) = 0\}$$

There exists $\epsilon > 0$ so that

$$\hat{h}_a(t) + \hat{h}_b(t) \geq \epsilon$$

for all but finitely many $t$ (and in particular, $|\mathrm{Tor}(a) \cap \mathrm{Tor}(b)| < \infty$)

Compare: Szpiro, Ullmo, Zhang
proof of the Bogomolov Conjecture

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The intersection $\mathrm{Tor}(a) \cap \mathrm{Tor}(b)$ is infinite if and only if $a = b$.

# Three ingredients in the dynamical proof

1. Infinite torsion sets: bifurcations + Montel's Theorem (1920)

   For every $a$, the set $\mathrm{Tor}(a) = \{t : P_t \text{ is torsion on } E_t\}$ is infinite.

2. Equidistribution theorem for points of small height on $P^1$
(Baker--Rumely, Favre--Rivera-Letelier, Chambert-Loir, 2006)

   For **algebraic** $a \neq 0, 1$, the set $\mathrm{Tor}(a)$ (or any infinite Galois invariant subset) is uniformly distributed with respect to a canonical measure $\mu_a$ on $\mathbb{C} \setminus \{0,1\}$.

3. A study of the bifurcation measure

   $\mu_a = \mu_b$ if and only if $a = b$

Compare:
Baker-D. 2011
Yuan-Zhang 2011
Ghioca-Hsia-Tucker 2012

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Hardest part:
estimates to show hypotheses
are satisfied (need continuous
potentials at singularities).

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# Equidistribution on $X = \mathbb{C} \setminus \{0, 1\}$

Let $k$ be a number field. Fix $a \in k(t)$, with $a \neq 0, 1, t$.

height function on $X$: $\quad \hat{h}_a(t) := \hat{h}_{E_t}(P_t)$

Take any infinite sequence of parameters $t_n$ where $P_{t_n}$ is torsion on $E_{t_n}$. (Then $\hat{h}_a(t_n) = 0$ for all $n$.) Let $G = \mathrm{Gal}(\bar{k}/k)$.

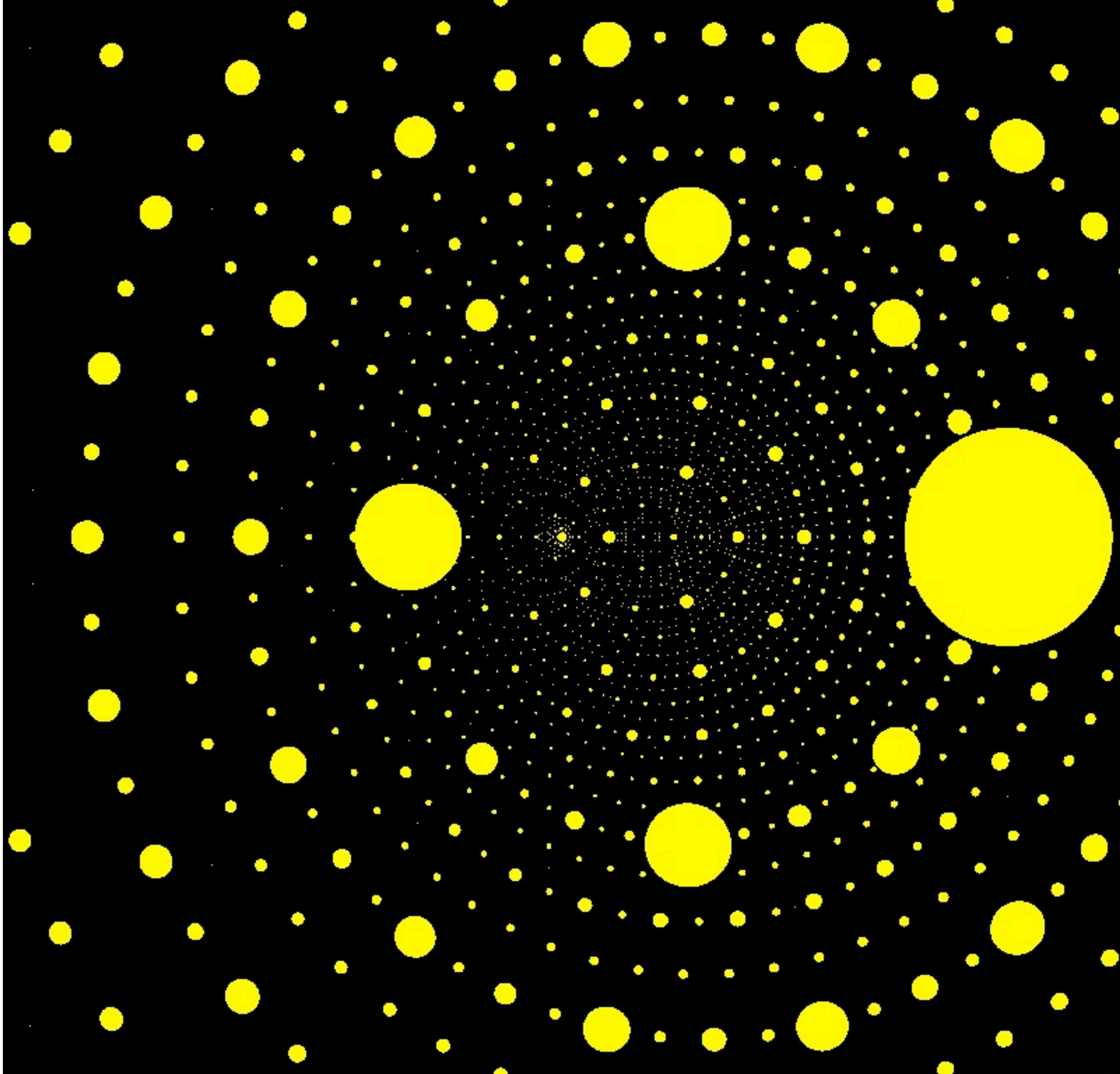
$$\mu_n = \frac{1}{|G \cdot t_n|} \sum_{t \in G \cdot t_n} \delta_t$$

converge (in the weak-$*$ topology) to the bifurcation measure $\mu_a$ on $\mathbb{P}^1(\mathbb{C})$.

(In fact, the measures converge to a probability measure $\mu_{a,v}$ on the Berkovich projective line $\mathbf{P}^1_{\mathbb{C}_v}$ for each place $v$ of $k$. The measure $\mu_{a,v}$ is the Laplacian of the local height function. This works for any sequence $t_n$ with $\hat{h}_a(t_n) \to 0$ as $n \to \infty$.)

$a = 2$

Plot: parameters $t$ where $a$ is the $x$-coordinate of a torsion point on $E_t$, of order $2^n$ with $n < 8$.



$-3 < \operatorname{Re} t < 5$

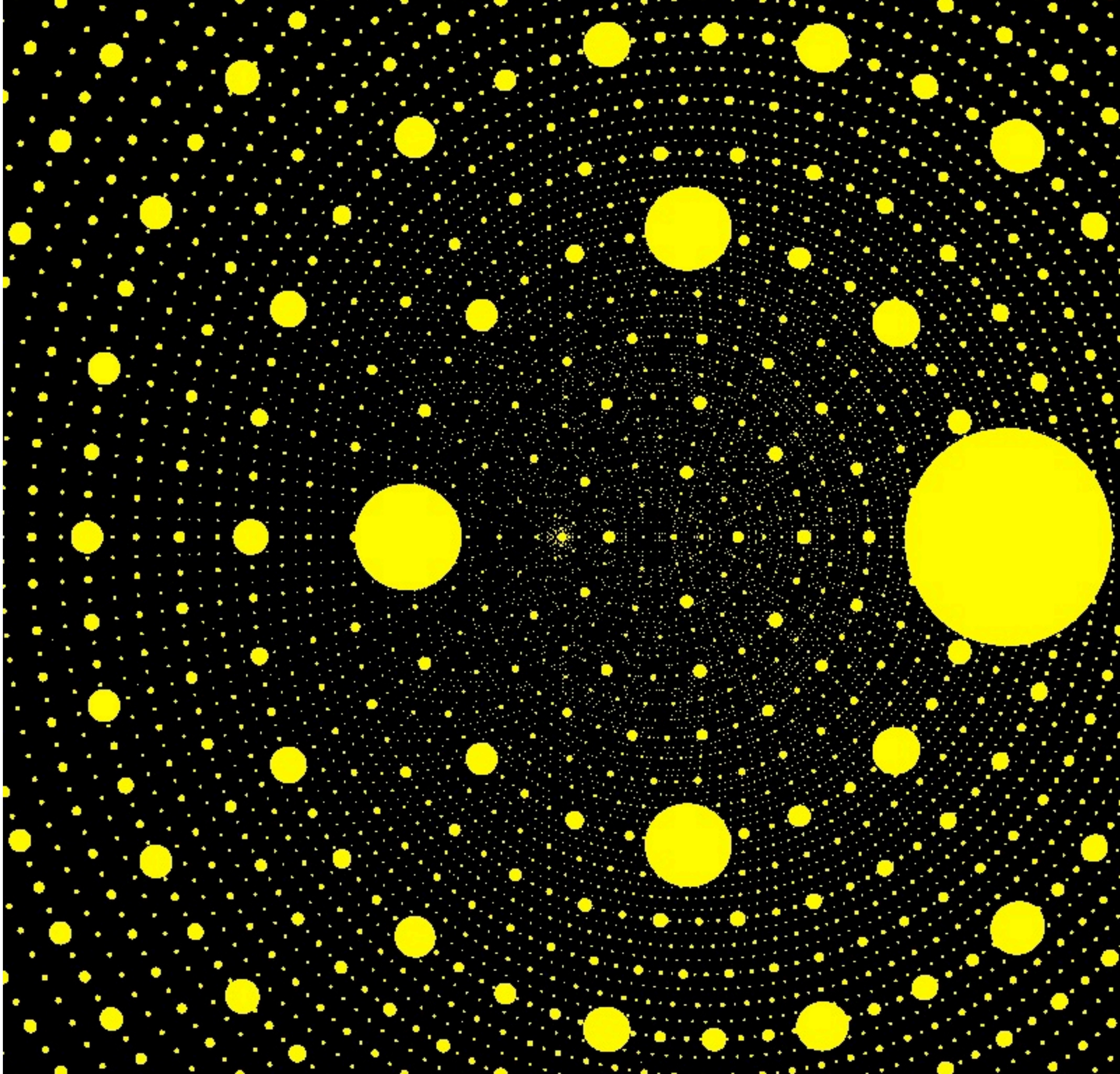
$-4 < \operatorname{Im} t < 4$

$a = 2$

Plot: parameters $t$ where $a$ is the $x$-coordinate of a torsion point on $E_t$, of order $2^n$ with $n < 10$.

$-3 < \operatorname{Re} t < 5$

$-4 < \operatorname{Im} t < 4$

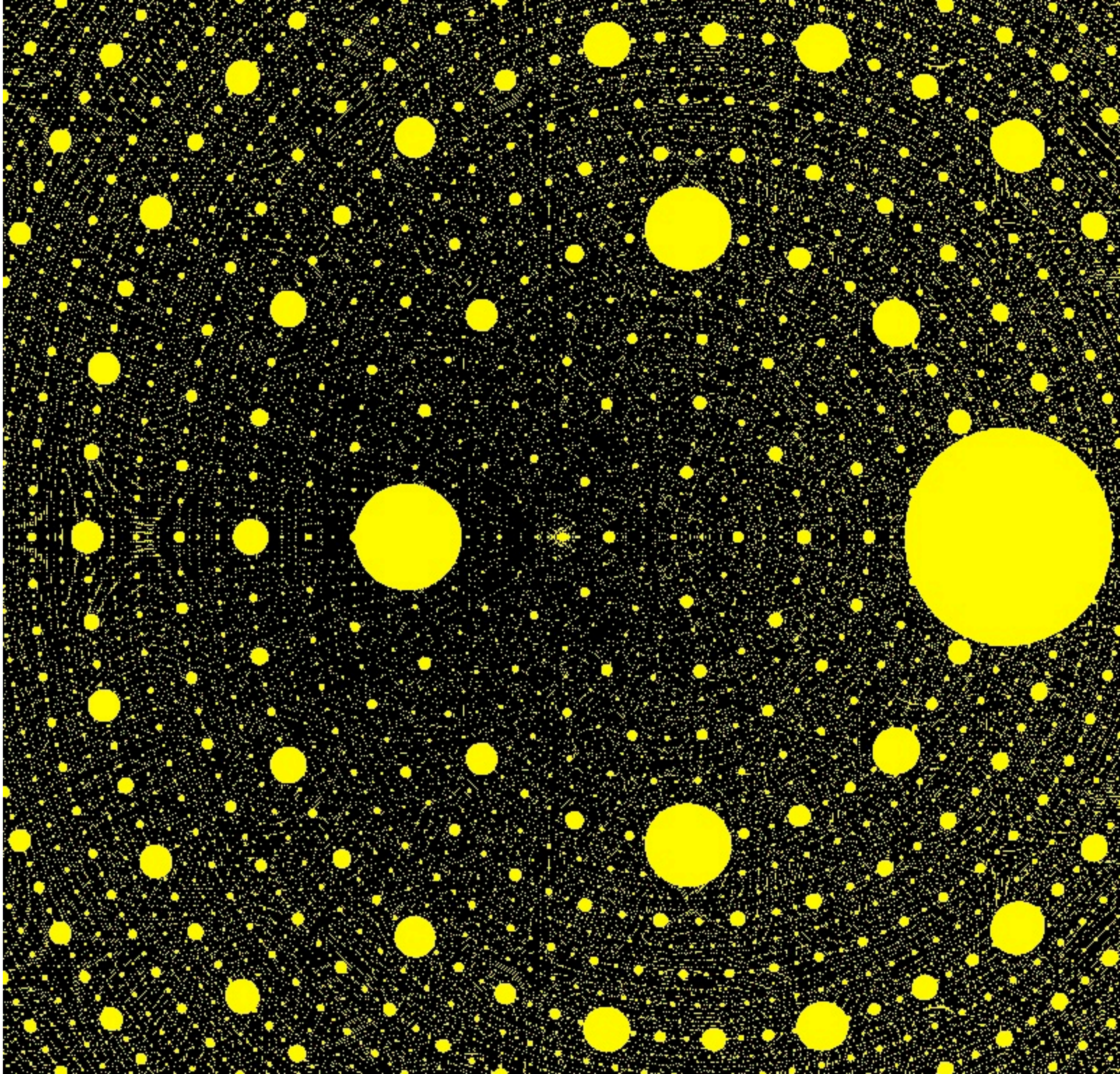


$a = 2$

Plot: parameters $t$ where $a$ is the $x$-coordinate of a torsion point on $E_t$, of order $2^n$ with $n < 15$.

$-3 < \operatorname{Re} t < 5$

$-4 < \operatorname{Im} t < 4$

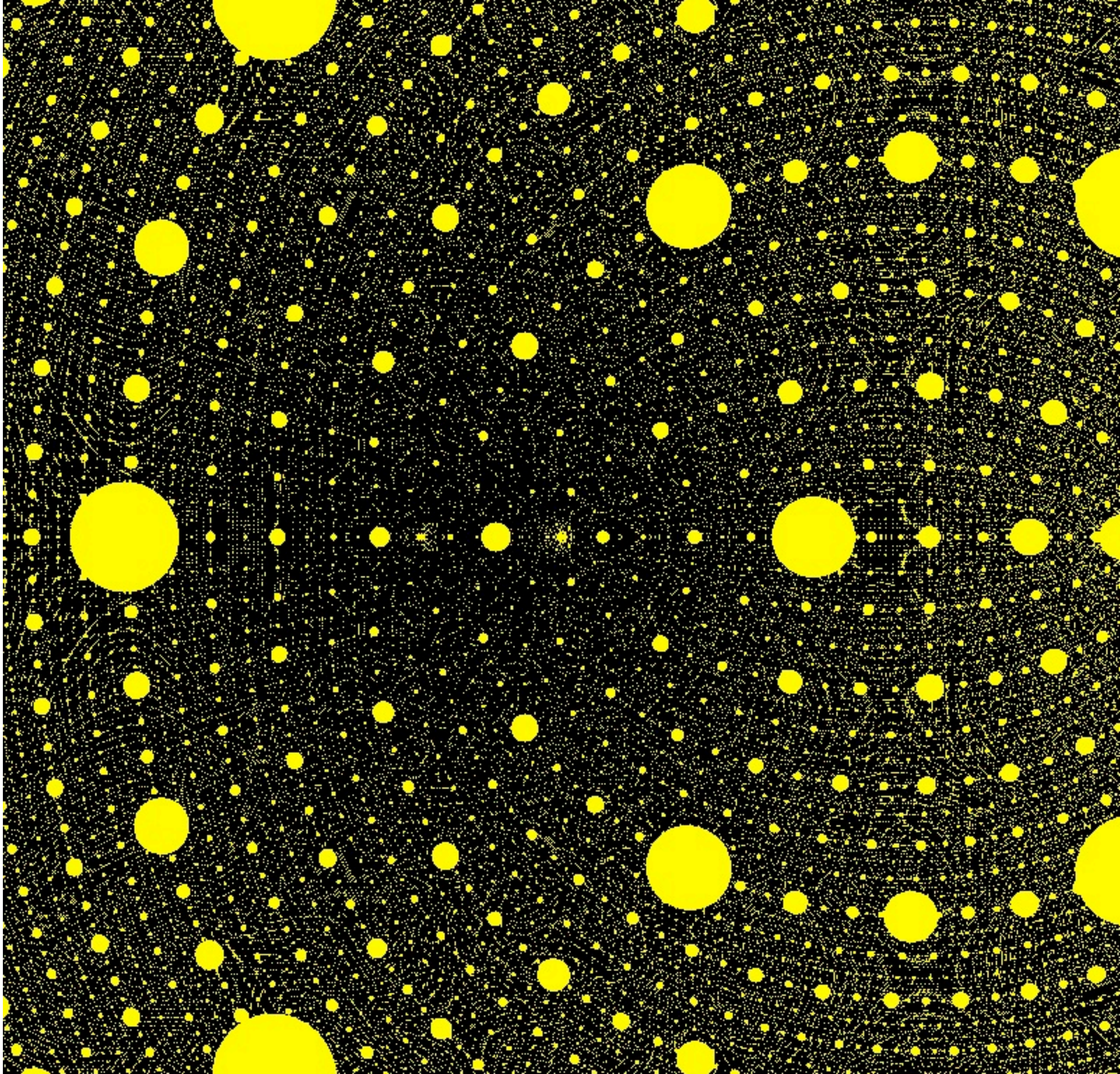


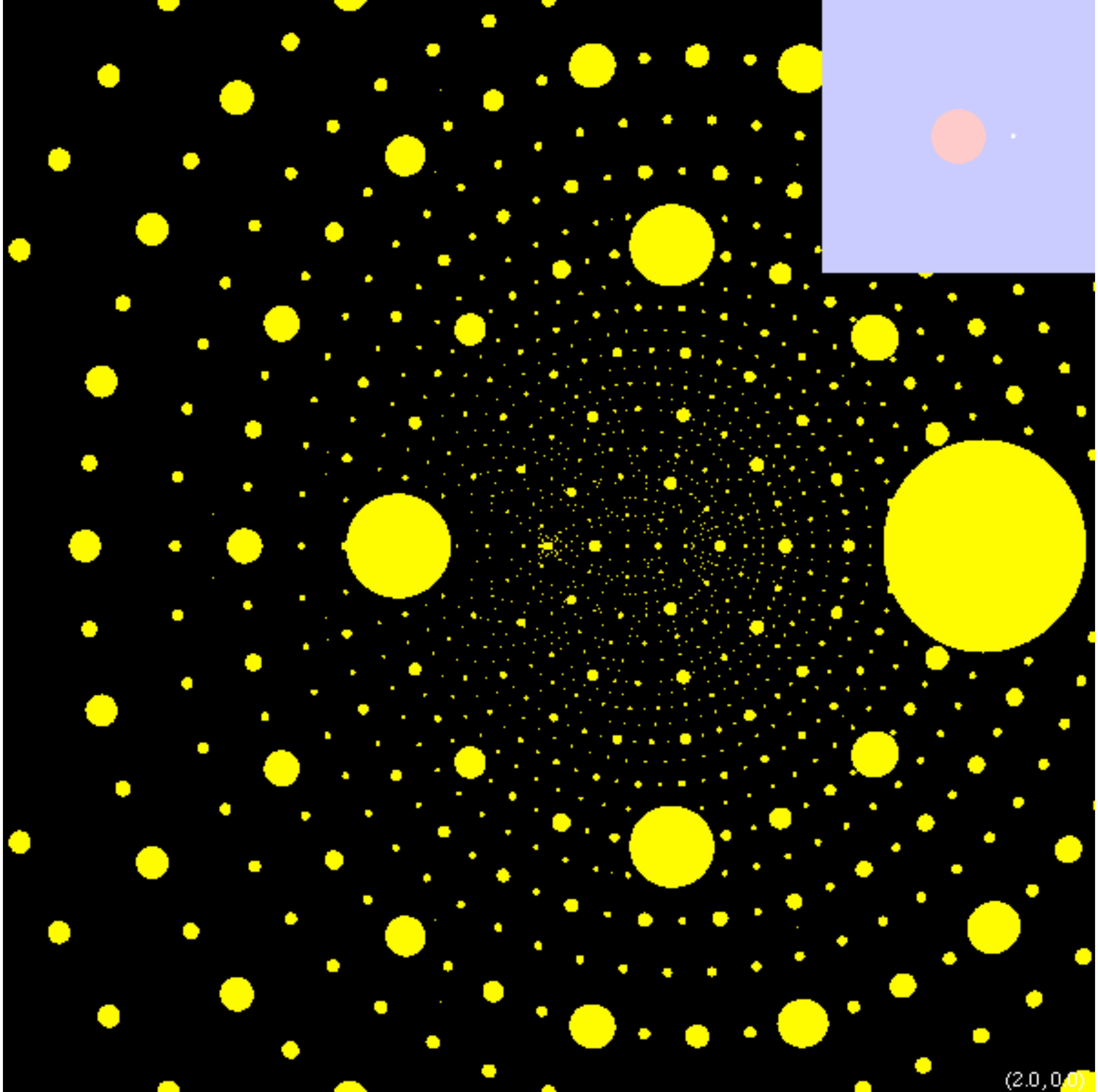
$a = 5$

Plot: parameters $t$ where $a$ is the $x$-coordinate of a torsion point on $E_t$, of order $2^n$ with $n < 15$.

$-3 < \operatorname{Re} t < 5$

$-4 < \operatorname{Im} t < 4$




(2.0,0.0)

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There are only finitely many $t \in \mathbb{C} \setminus \{0,1\}$ for which both $P_t$ and $Q_t$ are torsion points on $E_t$.

$$\Updownarrow$$

$a \neq b \in \mathbb{C} \setminus \{0,1\}$

More generally:
for $a, b \in \overline{\mathbb{C}(t)}$,
assume $nP \neq mQ$
for all $n, m \in \mathbb{Z} \setminus \{0\}$

# Simultaneously preperiodic

**Theorem.** Let

$$f_t(z) = \frac{(z^2-t)^2}{4z(z-1)(z-t)}$$

be the degree-4 Lattès family of rational functions. Fix $a \neq b$ in $\mathbb{C} \setminus \{0,1\}$. Then there are finitely many parameters $t$ for which both $a$ and $b$ are preperiodic.

More generally:
there are no maps $g_t$, $h_t$,
commuting with $f_t$ for all $t$,
so that $g(a) \equiv h(b)$.

**Zannier's Question.** Fix any one-parameter family of rational functions $\{f_t,\ t \in X\}$ and two points $a, b : X \to \mathbb{P}^1$. If $a(t)$ and $b(t)$ are simultaneously preperiodic for infinitely many parameters $t \in X$, what can we conclude about $a$ and $b$?

## with Matt Baker (2013):

**Conjecture.** Let $V$ be an $N$-dimensional complex algebraic variety in the moduli space $M_d$ of rational maps of degree $d$. Let $(a_0, a_1, \ldots, a_N)$ be an $(N+1)$-tuple of marked points. Then the points are *simultaneously preperiodic* on a Zariski-dense subset of $V$ if and only if the points are dynamically related.

A collection of $n$ points $a_1, \ldots, a_n$ is **dynamically related** along $V$ if there exists a subvariety $X \subset (\mathbb{P}^1_k)^n$, with $k = k(V)$ such that

(1) $(a_1, \ldots, a_n) \in X$, and

(2) $X$ is forward-invariant under $(f, \ldots, f)$

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Goal 1: show equidistribution of these special parameters $t$.

Goal 2: analyze the bifurcation measures $\mu_a$ and $\mu_b$.

Goal 3: If $\mu_a = \mu_b$ then how are $a$ and $b$ related?

Recall:

$$U(t) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} \log |f_t^n(P(t))|$$

$$\mu_P = \Delta U$$

**Theorem.** (D. 2015) If $\mu_P = 0$ on $X$ then $P$ is preperiodic for $f$.

## Special case: when the points are critical points

Let $f_t$ be a 1-parameter family of polynomials of degree $d \geq 2$.
Assume the critical points $c_i(t)$ are *polynomial* in $t$, $i = 1, \ldots, d-1$.

**Theorem.** (Baker-D., 2013) The following are equivalent:

(1) the polynomial $f_t$ is PCF for infinitely many $t$

(2) every pair of active critical points $c_i, c_j$ satisfies a critical orbit relation,

$$f_t^n(c_i(t)) = h_t(f_t^m(c_j(t)))$$

where $h \in \mathbb{C}[t, z]$ commutes with an iterate $f_t^l$ for all $t$.

Ingredient 1: an arithmetic equidistribution theorem in the Berkovich projective line (Baker-Rumely, Favre-Rivera-Letelier, Chambert-Loir, 2006)

Ingredient 2: classical complex analysis, univalent function theory, Ritt's decomposition theory (1925), Medvedev-Scanlon (2012)

## Higher dimensional parameter spaces

$f_t, t \in X$, a family of rational functions
$P : X \to \hat{\mathbb{C}}$ holomorphic

Bifurcation measure on a Riemann surface $\longrightarrow$ Bifurcation currents on a complex manifold

$$\mu_P = \Delta U_P$$

$$U(t) = \lim_{n\to\infty} \frac{1}{(\deg f)^n} \log |f_t^n(P(t))|$$

$$T_P = \partial\bar{\partial} U_P$$

$$T_P^k = (\partial\bar{\partial} U_P)^{\wedge k}$$

$$k \leq \dim_{\mathbb{C}} X$$

**Question.** If $T_P^k = T_Q^k$ for some $k$, what can we conclude about $P$ and $Q$? Do their orbits coincide under iteration of $f_t$?